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EQUIVALENT

MARKOV-RENEWAL PROCESSES

by

Burton Simon

Department of Industrial Engineering and Operations Research
Virginia Polytechnic Institute and State University
Blacksburg, Virginia

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The concept of strong and weak lumpability between Markov chains was introduced by Burke and Rosenblatt in 1958. In 1969 Serfozo showed that the concept of lumpability extends easily to Markov-renewal processes (MRP's). These concepts are apparently considered unimportant by the masses since there has been very little reference to them in the literature since 1972. The reason for the lack of interest is probably that the conditions for strong lumpability are too strong to be useful and nobody has ever considered the

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important special case of weak lumpability from a MRP to a renewal process. What is shown here is that in an appropriate modified form, these concepts are important in both application and in the foundational study of MRP's.

Equivalence and collapsibility between MRP's are defined, and necessary, sufficient, and necessary and sufficient conditions are given for them. It is shown that equivalence, collapsibility, weak lumpability and strong lumpability are morphisms between MRP's, and their relations to one another are examined.

Equivalence between a MRP and a renewal process is examined in detail. Specific results are obtained for irreducible, reducible, periodic and transient MRP's. These results are applied to problems concerning flows in queueing networks. It is shown that several well known results in queueing theory are examples of equivalence (for instance Burke's Theorem). New and simpler proofs are given for them. Some questions, previously unresolved, are answered using the techniques developed here; most notably the question of when the input process to the M/M/1 queue with instantaneous Bernoulli feedback is renewal.

Convolutions of MRP's are examined, and conditions are given for equivalence to be preserved under convolution. It is also shown that an important class of Markov-renewal equations can be simplified if the underlying MRP is equivalent to the renewal process.

Finally, it is shown that the ideas developed here can be extended to MRP's on general state spaces. The definitions of equivalence, collapsibility, weak lumpability and strong lumpability are given in the general setting. Examples from queueing theory that would make use of the generalized results are given.

OUTLINE OF MAIN RESULTS

I. Equivalence

A. from a MRP to a renewal process:

1. general MRP: Thm's 2.2.1, 2.2.2, 2.2.7, 2.5.1, 2.5.2
2. finite MRP: Thm's 2.2.8, 2.2.9
3. irreducible MRP: Thm's 3.1.1, 3.1.2
4. reducible MRP: Thm's 3.2.1, 3.2.3, 3.2.4
5. periodic MRP: Thm 3.3.1
6. transient MRP: Thm's 3.4.1, 3.4.2
7. preservation under convolution: Thm's 4.2.1, 4.2.2, 4.3.1
8. Markov-renewal equations: Thm's 4.3.3, 4.3.4

B. from a MRP to a MRP: Thm's 2.7.1, 5.4.4

II. Other Morphisms (section 5.4)

A. collapsibility of a MRP to a MRP:

1. general MRP's: Thm's 2.6.1, 2.6.2, 2.6.3, 2.6.4, 2.6.5, 2.6.6, 5.4.3
2. reducible MRP's: Thm 3.2.2
3. periodic MRP's: Thm's 3.3.2, 3.3.3

B. weak lumpability: Thm's 2.2.1, 5.4.2 (known results in section 1.2)

C. strong lumpability: Thm's 2.2.2, 5.4.1 (known results in section 1.2)

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CHAPTER I

INTRODUCTION

1. Introduction

The main purpose of this work is to introduce some new concepts in Markov-renewal theory which are of theoretical and practical interest.

A Markov-renewal process (MRP) is a sequence of pairs of random variables $\{X_n, T_n\}$ where X_n takes values in some measure space $(S, \mathscr{A})$ and T_n takes values in $[0,\infty]$, that satisfies

$$P(X_{n+1} \in A,\ T_{n+1} \le t \mid X_n, \cdots, X_0,\ T_n, \cdots, T_1) = P(X_{n+1} \in A,\ T_{n+1} \le t \mid X_n) \text{ a.s.}$$

for $n = 1,2,\cdots$, $A \in \mathscr{A}$, $t \in [0,\infty]$. It is assumed that there is a transition function $Q_{xA}(t) = P(X_{n+1} \in A,\ T_{n+1} \le t \mid X_n = x)$ which is called the kernel of the MRP. It is clear that the kernel and the initial distribution γ (i.e. $\gamma(A) = P(X_0 \in A)$) completely specifies the distribution of the process $\{X_n, T_n\}$.

It would seem that one of the basic questions in Markov-renewal theory would be to determine when two MRP's with different characterizations (i.e. different state space, kernel, or initial distribution) are "the same". A review of the literature shows that this question has been largely ignored. Actually, the only work done in this area has been the extension of the concept of strong and weak lumpability between Markov Chains to strong and weak lumpability between MRP's. A description and review of those results will be given in the next section.

Even the present knowledge of strong and weak lumpability

between MRP's is quite incomplete since no attempt is ever made to apply these ideas or to see if and when they occur in practice.

Consider the departure process from an M/M/1/N queue ($N \leq \infty$). It can be shown ([4], for instance) that the departure process is most naturally characterized as a MRP with $N+1$ states (corresponding to the length of the queue the instant after a departure). When $N = \infty$ (i.e. the M/M/1 queue), a countably infinite state MRP characterizes the departure process. It is well known, though, [2], [4], that the departure process from a steady state M/M/1 queue is a Poisson process, which like any renewal process is a one state MRP.[1] This is a nontrivial example of a MRP with two different characterizations. Several other similar examples can be found (see section 2.4).

Attempting to construct definitions for two MRP's being "the same" leads to the construction of morphisms between MRP's. Four morphisms will be defined; strong lumpability, weak lumpability, collapsibility and equivalence. Although strong and weak lumpability have never been defined outside the realm of finite state MRP's, there is no problem extending their definitions to the general case.

The conditions for strong lumpability are very restrictive and although they can be written down in a mathematically attractive form, strong lumpability is not as interesting a morphism as the others. Weak lumpability is a more interesting morphism, but the mathematics

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[1] A renewal process is a one state MRP since there is only one kind of event. Sometimes the total number of renewals up to time t (the counting process) is called the state of the renewal process. In that context the state of a renewal process can take values in the nonnegative integers. In our context though, the state of a MRP at the time of an event specifies the distribution of the time until the next event. Thus, a renewal process need have only one state since those times are i.i.d. random variables.

describing it is not very pleasant. Strong lumpability always implies weak lumpability but the converse is not true.

From a mathematical point of view, collapsibility is the most pleasing of the four morphisms. It is easy to show that weak lumpability implies collapsibility but the converse is presently unresolved (due to the complexity of weak lumpability). Strong evidence is given in section 2.6 towards the belief that collapsibility is a weaker condition than weak lumpability.

The weakest of the morphisms is equivalence. From a practical point of view, though, it is the most important. Equivalence is exactly the condition that allows one to substitute one MRP for another in (say) a queueing network without changing any of the important aspects of the system. It is easy to show that equivalence is weaker than the other three morphisms. It is also easy to show that in the important special case where one of the MRP's is a renewal process, weak lumpability collapsibility and equivalence are identical.

The precise definitions of these morphisms in the general context is given in chapter 5. The definitions for the finite and countable case are in chapters 1 and 2.

The vast majority of the present work is concerned with the simplest cases of the morphisms (between a MRP and a renewal process, and between two finite or countable MRP's). The reason for this is twofold. First of all, as far as applications are concerned, these are the most important cases. Also, since so little work has been done in this area it seems rediculous to jump into the general setting without a thorough discussion of the important special cases.

2. Lumpability

Probably the simplest case of equivalence between MRP's is lumpability in Markov chains [8]. Let $\{X_n\}$ be a Markov chain on a finite or countable state space S. Let $A_1, A_2, \cdots, A_n$ be a partition of S, and let $F: S \to \{A_1, A_2, \cdots, A_n\}$ be the map that "lumps" the state space S onto the partition $\{A_1, A_2, \cdots, A_n\}$. The process $\{F(X_n)\}$ may or may not be a Markov chain. In general, the probability of going from A_i to A_j in $\{F(X_n)\}$ will depend on precisely which element of A_i the $\{X_n\}$ process is in. If for each i and j, though, the probability of going from A_i to A_j is independent of the state in A_i that the $\{X_n\}$ process is in, then the process $\{F(X_n)\}$ is a Markov chain. When this happens we say $\{X_n\}$ is strongly lumpable to $\{F(X_n)\}$. This is a special case of the equivalence to be defined.

For example, say $S = \{1,2,3\}$ and let $\{X_n\}$ have transition probability matrix

$$\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \end{pmatrix} .$$

Let $F(1) = A_1$, $F(2) = F(3) = A_2$. The process $\{F(X_n)\}$ is a Markov chain on $\{A_1, A_2\}$ with transition probability matrix

$$\begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} .$$

If $\{X_n\}$ is strongly lumpable to $\{F(X_n)\}$ then no matter which state in S the process starts in, $\{F(X_n)\}$ will be a Markov chain. In fact, even if the precise state that the process begins in is not known, the ensueing $\{F(X_n)\}$ process is a Markov chain.

Sometimes, even though $\{X_n\}$ is not strongly lumpable to $\{F(X_n)\}$, the process $\{F(X_n)\}$ is a Markov chain when $\{X_n\}$ is in steady state. When this happens we say $\{X_n\}$ is weakly lumpable to $\{F(X_n)\}$.

If S has m elements ($m < \infty$) and $F(S)$ has n elements $A_1, A_2, \cdots, A_n$ ($n < m$), then the following m x n matrix, $\underset{\sim}{U}$, can be constructed. Let

$$\underset{\sim}{U}_{ij} = \begin{cases} 0, & \text{if } i \notin A_j, \\ 1, & \text{if } i \in A_j. \end{cases}$$

There is a vector π that satisfies $\pi P = \pi$ (the steady state vector) where P is the transition probability matrix for $\{X_n\}$. Let Π be an n x m matrix with

$$\Pi_{ij} = \begin{cases} 0, & \text{if } j \notin A_i, \\ \dfrac{\pi_j}{\sum\limits_{k \in A_i} \pi_k}, & \text{if } j \in A_i. \end{cases}$$

The i^{th} row of Π is the conditional probability of being in state j given that the process is in steady state and that the process is in A_i.

Kemeny and Snell [8], show that $\{X_n\}$ is strongly lumpable to $\{F(X_n)\}$ if and only if $P\underset{\sim}{U} = \underset{\sim}{U}(\Pi P\underset{\sim}{U})$, and that if $\{F(X_n)\}$ is a Markov chain then its transition probability matrix is $\Pi P\underset{\sim}{U}$. They also show that $\Pi P = (\Pi P\underset{\sim}{U})\Pi$ or $P\underset{\sim}{U} = \underset{\sim}{U}(\Pi P\underset{\sim}{U})$ is a sufficient condition for $\{X_n\}$ to be weakly lumpable to $\{F(X_n)\}$. For example let $S = \{1,2,3\}$ and set

$F(1) = A_1$, $F(2) = F(3) = A_2$. Suppose the transition probability matrix for $\{X_n\}$ is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

In this case $\pi = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ so

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

$\{X_n\}$ is not strongly lumpable to $\{F(X_n)\}$ since

$$P(F(X_n) = A_1 | F(X_{n-1}) = A_2)$$

depends on whether X_{n-1} is equal to 2 or 3. This can be seen formally by noting that $P\underset{\sim}{U} \neq \underset{\sim}{U}(\Pi P\underset{\sim}{U})$. In steady state, though, $\{F(X_n)\}$ is a Markov chain since $\Pi P = (\Pi P\underset{\sim}{U})\Pi$. The resulting Markov chain $\{F(X_n)\}$ has a transition probability matrix

$$\Pi P\underset{\sim}{U} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The necessary conditions for weak lumpability are much less appealing than the necessary and sufficient condition for strong lumpability or the sufficient conditions for weak lumpability. If γ is a

probability vector on S then define $[\gamma]^i$ to be the vector of conditional probabilities of being in state j $(j = 1,2,\cdots,m)$, given that the process is in A_i. For example, the i^{th} row of the matrix Π is $[\pi]^i$. Let Γ_j be the set of all finite sequences of states in F(S) that end with A_j. If $A_{i_1}, A_{i_2},\cdots,A_{i_k}, A_j$ and $A_{j_1}, A_{j_2},\cdots,A_{j_q}, A_j$ are two elements of Γ_j then for $\{X_n\}$ to be weakly lumpable to $\{F(X_n)\}$ it must be true that for each $\alpha \varepsilon (1, 2,\cdots,n)$.

$$\sum_{i \varepsilon S} \sum_{\beta \varepsilon A_\alpha} P_{i\beta}\, \gamma_i^{\,1} = \sum_{i \varepsilon S} \sum_{\beta \varepsilon A_\alpha} P_{i\beta}\, \gamma_i^{\,2}$$

where

$$\gamma^1 = [\ \cdots\ [[[[\pi]^{i_1} P]^{i_2} P]^{i_3} \cdots P]^{i_k} P]^j$$

and

$$\gamma^2 = [\ \cdots\ [[[[\pi]^{j_1} P]^{j_2} P]^{j_3} P]^{j_4} \cdots P[^{q} P]^j \ .$$

Serfozo [16] showed that strong and weak lumpability can be defined for MRP's in an analogous manner. In fact, the conditions for strong and weak lumpability in MRP's are virtually indentical to the conditions for Markov chains. If $\{X_n,T_n\}$ is a MRP on a finite state space $S = \{1,2,3,\cdots,m\}$, with kernel Q(t)

$$(\text{i.e. } Q_{ij}(t) = P(X_{n+1} = j,\ T_{n+1} \le t \mid X_n = i))$$

and F: $S \to \{A_1,A_2,\cdots,A_n\}$ is a partition of the state space then $\{X_n,T_n\}$ is said to be strongly lumpable to $\{F(X_n),\ T_n\}$ if $\{F(X_n),\ T_n\}$ is a MRP.

Again, let π be the steady state vector for the embedded Markov chain (i.e. $\pi Q(\infty) = \pi$), and let Π, $\underset{\sim}{U}$ be defined as before. Serfozo shows that $\{X_n,T_n\}$ is strongly lumpable to $\{F(X_n),\ T_n\}$ if and only if

$Q(t)\underline{U} = \underline{U}(\Pi Q(t)\underline{U})$ for all $t \in [0,\infty]$. Likewise if for all t, $Q(t)\underline{U}$ $= \underline{U}(\Pi Q(t)\underline{U})$ or $\Pi Q(t) = (\Pi Q(t)\underline{U})\Pi$ then $\{F(X_n), T_n\}$ is a MRP in steady state (i.e. weakly lumpable). Unfortunately, the necessary conditions for weak lumpability are again very complicated. Let Γ_j be the set of all finite sequences of states in F(S) that end with A_j. If $A_{i_1}, A_{i_2}, \cdots, A_{i_k}, A_j$ and $A_{j_1}, A_{j_2}, \cdots, A_{j_q}, A_j$ are two elements of Γ_j and $\{t_1, t_2, \cdots t_k\}$, $\{s_1, s_2, \cdots, s_q\}$ are two sequences of positive real numbers then for $\{X_n, T_n\}$ to be weakly lumpable to $\{F(X_n), T_n\}$ it must be true that for each $\alpha \in (1,2,\cdots,n)$ and $t \in [0,\infty]$,

$$\sum_{i \in S} \sum_{\beta \in A_\alpha} Q_{i\beta}(t)\, \gamma_i^{\,1} = \sum_{i \in S} \sum_{\beta \in A_\alpha} Q_{i\beta}(t)\, \gamma_i^{\,2}$$

where

$$\gamma^1 = [\cdots [[[[\pi]^{i_1} Q(t_1)]^{i_2} Q(t_2)]^{i_3} Q(t_3)]^{i_4} \cdots Q(t_{k-1})]^{i_k} Q(t_k)]^{j}$$

and

$$\gamma^2 = [\cdots [[[[\pi]^{j_1} Q(s_1)]^{j_2} Q(s_2)]^{j_3} Q(s_3)]^{j_4} \cdots Q(s_{q-1})]^{j_q} Q(s_q)]^{j}.$$

In this paper a type of equivalence will be defined that includes all of the cases discussed so far and has the added property that a necessary and sufficient condition for two MRP's to be equivalent can be written in a simple form.

3. Definitions and Notation

In the first four chapters, all MRP's will have finite or countable state spaces. Let S be a finite or countable subset of $\mathbb{R}$, let $\mathcal{B}$ denote the Borel subsets of $[0,\infty]$ and let $\Omega = (S \times [0,\infty])^\infty$, the countable cartesian product of $S \times [0,\infty]$ with itself. Thus, $\omega \in \Omega$ can be represented by

$$\omega = \{(x_n(\omega),\ t_n(\omega)) : x_n(\cdot) \in S,\ t_n(\cdot) \in [0,\infty],\ n = 0,1,2,\cdots\}.$$

Let $X_n(\omega) = x_n(\omega)$ and $T_n(\omega) = t_n(\omega)$. Let $\mathcal{F}$ be the smallest σ-algebra on Ω that makes $\{X_n, T_n : n = 0,1,2,\cdots\}$ random variables, and let $\mathcal{F}_k$ be the smallest σ-algebra that makes $\{X_n, T_n : n = 0,1,2,\cdots,k\}$ random variables. Let P be a probability measure on $(\Omega, \mathcal{F})$.

Definition 1.3.1. The process $\{X_n, T_n\}$, $n = 0,1,2,\cdots$, is a Markov-renewal process (MRP) if there is a $Q_{ij}(t) : S \times S \times [0,\infty] \to [0,1]$ such that

$$\forall n,\quad P(X_n = j,\ T_n \le t \mid \mathcal{F}_{n-1}) = Q_{X_{n-1},j}(t) \quad \text{a.s.}$$

Definition 1.3.2. The kernel of a MRP is the matrix of sub-distribution functions, $Q(t) = \{Q_{ij}(t) : i,j \in S,\ t \in [0,\infty]\}$.

Lemma 1.3.1. If $\{X_n, T_n\}$ is a MRP with kernel $Q(t)$ then

$$P(X_0 \in A_0,\cdots,X_m \in A_m,\ T_1 \le t_1,\cdots,T_m \le t_m)$$

$$= \sum_{j_0 \in A_0} \cdots \sum_{j_n \in A_n} P(X_0 = j_0) Q_{j_0 j_1}(t_1) \cdots Q_{j_{m-1} j_m}(t_m).$$

Proof. Clearly $P(X_0 \in A_0) = \sum_{j_0 \in A_0} P(X_0 = j_0)$. Assume the hypothesis is true for $k \le m$. Then,

$$P(X_0 \in A_0,\cdots,X_m \in A_m,\ T_1 \le t_1,\cdots,T_m \le t_m)$$

$$= \sum_{j_m \in A_m} \sum_{j_{m-1} \in A_{m-1}} P(X_0 \in A_0,\cdots,X_{m-2} \in A_{m-2},$$

$$X_{m-1} = j_{m-1},\ X_m = j_m,\ T_1 \le t_1,\cdots,T_m \le t_m)$$

$$= \sum_{j_m \varepsilon A_m} \sum_{j_{m-1} \varepsilon A_{m-1}} P(X_m = j_m,\ T_m \leq t_m | X_{m-1} = j_{m-1},\ X_{m-2} \varepsilon A_{m-2},$$

$$\cdots, X_0 \varepsilon A_0,\ T_{m-1} \leq t_{m-1}, \cdots, T_1 \leq t_1)$$

$$\cdot P(X_{m-1} = j_{m-1},\ X_{m-2} \varepsilon A_{m-2}, \cdots, X_0 \varepsilon A_0,$$

$$T_{m-1} \leq t_{m-1}, \cdots, T_1 \leq t_1)$$

$$= \sum_{j_m \varepsilon A_m} \sum_{j_{m-1} \varepsilon A_{m-1}} P(X_m = j_m,\ T_m \leq t_m | X_{m-1} = j_{m-1})$$

$$\cdot \sum_{j_0 \varepsilon A_0} \sum_{j_{m-2} \varepsilon A_{m-2}} P(X_0 = j_0) Q_{j_0 j_1}(t_1) \cdots Q_{j_{m-2} j_{m-1}}(t_{m-1})$$

$$= \sum_{j_0\ A_0} \sum_{j_m\ A_m} P(X_0 = j_0) Q_{j_0 j_1}(t_1) \cdots Q_{j_{m-1} j_m}(t_m). \ \square$$

Let $\mathscr{A}$ be the semi algebra of sets of the form $(X_0 \varepsilon A_0, \cdots, x_n \varepsilon A_n, T_1 \leq t_1, \cdots, T_n \leq t_n)$. For $A \varepsilon \mathscr{A}$, let $\hat{P}(A) = P(A)$. Since P is the extension of $\hat{P}$ from $\mathscr{A}$ to $\mathscr{F}$ it is clear from the lemma that the kernel, $Q(t)$, and the initial distribution, $\gamma = \{\gamma_j : j \varepsilon S\}$ (where $\gamma_j = P(X_0 = j)$) completely specify the distribution of $\{X_n, T_n\}$, $n = 0,1,2,\cdots$. For this reason, throughout this paper a MRP will be specified by its kernel and initial distribution.

<u>Definition</u> 1.3.3. A MRP, $\{X_n, T_n\}$, $n = 0,1,2,\cdots$, with kernel $Q(t)$ and initial distribution γ will be denoted (Q,γ). A MRP with kernel $Q(t)$ and unspecified initial distribution will be denoted Q.

Occassionally, we will need to consider the individual sample paths of the MRP (Q,γ). This poses no problem since we can define

$(\Omega, \mathscr{F})$ as before and construct P by extending $\hat{P}$.

If S has only one element then $\Omega = [0,\infty]^{\infty}$, $\mathscr{F} = \mathscr{B}^{\infty}$ and $\mathscr{F}_k = \mathscr{B}^k$. Thus, if $\{X_n, T_n\}$, $n = 0,1,2,\cdots$, is a MRP with only one state

$$P(T_n \leq t \mid \mathscr{F}_{n-1}) = P(T_n \leq t) = r(t) \quad \text{a.s.}$$

Thus, $\{T_1, T_2, \cdots\}$ is a sequence of i.i.d. nonnegative random variables with common distribution r(t).

Definition 1.3.4. A MRP with one state will be called a renewal process. The kernel of a renewal process is a scalar function, r(t). Since there can be no confusion about the initial distribution of a MRP with only one state, a renewal process, $\{T_1, T_2, \cdots\}$ with $P(T_n \leq t) = r(t)$ will be denoted r.

Let (Q, γ) be a MRP. By renumbering the states, the kernel Q(t) can always be put into the following canonical form. (see Çinlar [3])

$$
(1.3.1) \quad Q(t) = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
A_1(t) & & & & & & & & \\ \hline
 & A_2(t) & & & & & & & \\ \hline
 & & \ddots & & & & & & \\ \hline
 & & & A_k(t) & & & & & \\ \hline
C_{11}(t) & C_{12}(t) & \cdots & C_{1k}(t) & B_1(t) & & & & \\ \hline
C_{21}(t) & C_{22}(t) & \cdots & C_{2k}(t) & D_{21}(t) & B_2(t) & & & \\ \hline
C_{31}(t) & C_{32}(t) & \cdots & C_{3k}(t) & D_{31}(t) & D_{32}(t) & B_3(t) & & \\ \hline
\vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \\ \hline
C_{\ell 1}(t) & C_{\ell 2}(t) & \cdots & C_{\ell k}(t) & D_{\ell 1}(t) & D_{\ell 2}(t) & D_{\ell 3}(t) & \cdots & B_{\ell}(t)
\end{array}\right]
$$

where the A's and B's are square. If $k = 1$ and $\ell = 0$ (i.e. $Q(t) = A_1(t)$), the MRP is called irreducible. Otherwise it is reducible. By renumbering the states of an irreducible MRP the kernel can always be put into the following canonical form

$$
(1.3.2) \quad Q(t) = \left[\begin{array}{c|c|c|c|c}
 & Q_1(t) & & & \\ \hline
 & & Q_2(t) & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & Q_{n-1}(t) \\ \hline
Q_n(t) & & & &
\end{array}\right] .
$$

where the columns of $Q_j(t)$ correspond to the rows of $Q_{j+1}(t)$ (interpret $j+1 = 1$ if $j = n$). If $n = 1$ (i.e. $Q(t) = Q_1(t)$), the MRP is called aperiodic, otherwise it is periodic with period n.

Notation. U will always denote a column vector of 1's.

Definition 1.3.5. If $Q(t)$ is a kernel of a MRP then

$Q(\infty) = \lim_{t\to\infty} Q(t)$.

Since $Q(t)$ is right continuous it is possible that for some i, $\sum_j Q_{ij}(\infty) < 1$. This means that if $X_{n-1} = i$ there is a positive probability that $T_n = \infty$. This can be interpreted as a process that terminates at time $T = T_1 + T_2 + \cdots + T_{n^*}$ where $n^*(\omega) = \inf_n \{T_{n+1}(\omega) = \infty\}$. It is well known (Çinlar [3]) that if Q is irreducible, $P(n^* = \infty) = 0$ if $Q(\infty)U < U$ and $P(n^* = \infty) = 1$ if $Q(\infty)U = U$.

Definition 1.3.6. An irreducible MRP is said to be transient if $P(n^* = \infty) = 0$ and presistent if $P(n^* = \infty) = 1$. Note that a transient MRP and a MRP with transient states are different concepts. For example, if $r(\infty) = 1$ then

$$\begin{array}{|c|c|c|c|}
\hline
 & r(t) & & \\
\hline
 & & r(t) & \\
\hline
 & & & \ddots \\
\hline
 & & & \\
\hline
\end{array}$$

is not a transient MRP even though every state is transient.

In this paper irreducible MRP's will be persistent unless stated otherwise.

It is well known (Çinlar [3]) that if $Q(\infty)$ is an irreducible matrix there is a unique positive eigenvector associated with the larg-

est eigenvalue of $Q(\infty)$.

Definition 1.3.7. Let Q be an irreducible MRP. Let π satisfy $\pi Q(\infty) = \lambda\pi$ and $\pi U = 1$ where λ is the largest positive eigenvalue of $Q(\infty)$. π will be called the steady state vector for Q.

4. Elementary Facts about MRP's

Let (Q,γ) be a MRP. We will need to work with equations involving terms like

(1) $P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n, X_n = j)$

(2) $P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n | X_0 = i)$

(3) $P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n)$.

Fortunately, from lemma 1.3.1, these quantities can be written down in an attractive form using the matrices $\{Q(t)\}$, $t \in [0,\infty]$, and the vector γ.

(1') $P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n, X_n = j) = (\gamma Q(t_1)Q(t_2)\cdots Q(t_n))_j$

(2') $P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n | X_0 = i) = (Q(t_1)Q(t_2)\cdots Q(t_n)U)_i$

(3') $P(T_1 \leq t_1, T_2 \leq t_2, \cdots, T_n \leq t_n) = \gamma Q(t_1)Q(t_2)\cdots Q(t_n)U$.

Let $Q(t)$ be the kernel of a finite state, irreducible, aperiodic (possibly transient) MRP. By the Perron-Frobenius Theorem [3] there is a unique largest eigenvalue of $Q(\infty)$ which is positive, and that eigenvalue has an associated left and right eigenvector which have positive elements. Let λ be the eigenvalue, π the left eigenvector and β the right eigenvector. Assume that π is normalized so that $\pi U = 1$ and β is normalized so that $\pi\beta = 1$. Since λ has multiplicity one, we can write $Q(\infty) = AJA^{-1}$ where

$$J = \begin{bmatrix} \lambda & & & & \\ & J_2 & & & \\ & & J_3 & & \\ & & & \ddots & \\ & & & & J_k \end{bmatrix}, \quad A = \begin{bmatrix} \beta & B \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \pi \\ \hline C \end{bmatrix}.$$

Since λ is the largest eigenvalue,

$$\lim_{n\to\infty} \lambda^{-n} J^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$\lim_{n\to\infty} \lambda^{-n} Q^n(\infty) = A(\lim_{n\to\infty} \lambda^{-n} J^n) A^{-1} = \beta\pi.$$

In the case where (Q,γ) is persistent, $\lambda = 1$, $\beta = U$, and we get the well known result that $\lim_{n\to\infty} Q^n(\infty) = U\pi$.

If $Q(\infty)$ has period m (i.e. has form 1.3.2) then

$$Q^m(\infty) = \begin{bmatrix} \hat{Q}_1 & & & \\ & \hat{Q}_2 & & \\ & & \ddots & \\ & & & \hat{Q}_m \end{bmatrix},$$

where $\hat{Q}_j = Q_j Q_{j+1} \cdots Q_m Q_1 \cdots Q_{j-1}$. Each $\hat{Q}_j$ is irreducible and has the same maximal eigenvector λ. Thus if π_j and β_j are the normalized left and right eigenvectors associated with λ then

$$\lim_{n\to\infty} \lambda^{-nm} Q^{nm}(\infty) = \begin{bmatrix} \beta_1\pi_1 & & & \\ & \beta_2\pi_2 & & \\ & & \ddots & \\ & & & \beta_m\pi_m \end{bmatrix}$$

Furthermore, if $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_m)^T$, then

$$\lim_{n\to\infty} \frac{1}{m} \sum_{k=1}^{m} \lambda^{-(nm+k)} Q^{nm+k}(\infty) = \beta\pi.$$

Let Q have the general form (1.3.1) and assume now that $Q(\infty)U = U$. Say A_j has period m_j and let m be the least common multiple of the m_j's. Since each B_j is a transient sub MRP, we get

$$\lim_{n\to\infty} \frac{1}{m} \sum_{k=1}^{m} Q^{nm+k}(\infty) = \left[\begin{array}{cccc|c} U\pi_1 & & & & \\ & U\pi_2 & & & \\ & & \ddots & & 0 \\ & & & U\pi_n & \\ \hline p_{11}\pi_1 & p_{12}\pi_2 & \cdots & p_{1n}\pi_n & \\ p_{21}\pi_1 & p_{22}\pi_2 & \cdots & p_{2n}\pi_n & \\ \vdots & \vdots & & \vdots & 0 \\ p_{d1}\pi_1 & p_{d2}\pi_2 & \cdots & p_{dn}\pi_n & \end{array}\right]$$

where p_{ij} is the probability of ending up in A_j given the process starts in the i^{th} transient state. (see [3]).

CHAPTER II

EQUIVALENCE

1. Equivalence

Associated with each MRP (Q,γ) is a sequence of nonnegative (possibly extended valued) random variables $\{T_1,T_2,\cdots\}$. The distribution of $\{T_1,T_2,\cdots\}$ is given by

$$P(T_1 \le t_1,\ T_2 \le t_2,\cdots,T_n \le t_n) = \gamma Q(t_1)Q(t_2)\cdots Q(t_n)U.$$

Lemma 2.1.1. $\{T_1,T_2,\cdots\}$ is a sequence of i.i.d. nonnegative random variables (i.e. a renewal process) if and only if

$$\forall n,t_1,t_2,\cdots,t_n,\ \gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = (\gamma Q(t_1)U)(\gamma Q(t_2)U)\cdots(\gamma Q(t_n)U).$$

Proof. ($\Longrightarrow$) If $T_1,T_2,\cdots$ is a renewal process then

$$P(T_1 \le t_1,\ T_2 \le t_2,\cdots,T_n \le t_n) = P(T_1 \le t_1)P(T_1 \le t_2)\cdots P(T_1 \le t_n).$$

But this says $\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = (\gamma Q(t_1)U)(\gamma Q(t_2)U)\cdots(\gamma Q(t_n)U)$.

($\Longleftarrow$) If $\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = (\gamma Q(t_1)U)(\gamma Q(t_2)U)\cdots(\gamma Q(t_n)U)$ then $P(T_1 \le t_1,\ T_2 \le t_2,\cdots,T_n \le t_n) = P(T_1 \le t_1)P(T_1 \le t_2)\cdots P(T_1 \le t_n)$.

This leads to the definition of equivalence between a MRP and a renewal process.

Definition 2.1.1. Let (Q,γ) be a MRP and let r be a renewal process. (Q,γ) is equivalent to r (written $(Q,\gamma) \sim r$) if

$$\forall n,\ t_1,t_2,\cdots t_n,\ \gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n).$$

If for any initial distribution β, $(Q,\beta) \sim r$ then we write $Q \approx r$. If there exists an initial distribution β, such that $(Q,\beta) \sim r$ then $Q \sim r$.

Clearly, if $(Q,\gamma) \sim r$ it must be true that $r(t) = \gamma Q(t)U$. The main questions that will be answered in Chapters II and III are

(1) When is $(Q,\gamma) \sim r$?

(2) For a given MRP Q, when is it impossible to find an initial distribution γ, and a renewal process r that yields $(Q,\gamma) \sim r$?

(3) If $(Q,\gamma) \sim r$ and $(Q,\beta) \sim f$ when must $r = f$?

(4) If $(Q,\gamma) \sim r$ and $(Q,\beta) \sim r$ when must $\gamma = \beta$?

2. Conditions for Equivalence

In this section we give sufficient, necessary, and necessary and sufficient conditions for $(Q,\gamma) \sim r$.

Theorem 2.2.1. Let (Q,γ) be a MRP and let r be a renewal process. If $\forall t$, $\gamma Q(t) = r(t)\gamma$ then $(Q,\gamma) \sim r$.

Proof. If $\forall t$, $\gamma Q(t) = r(t)\gamma$ then $\forall n$, $t_1, t_2, \cdots, t_n$,

$$\begin{aligned}
\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U &= r(t_1)\gamma Q(t_2)Q(t_3)\cdots Q(t_n)U \\
&= r(t_1)r(t_2)\gamma Q(t_3)\cdots Q(t_n)U \\
&\ \ \vdots \\
&= r(t_1)r(t_2)\cdots r(t_n)\gamma U \\
&= r(t_1)r(t_2)\cdots r(t_n). \ \square
\end{aligned}$$

Theorem 2.2.2. If $\forall t$, $Q(t)U = Ur(t)$ then $Q \approx r$.

Proof. If $\forall t$, $Q(t)U = Ur(t)$ then $\forall n$, $t_1, t_2, \cdots t_n$

$$\beta Q(t_1)Q(t_2)\cdots Q(t_n)U = \beta U r(t_1)r(t_2)\cdots r(t_n) = r(t_1)r(t_2)\cdots r(t_n)$$

for any initial distribution β. $\square$

Theorem 2.2.1 and 2.2.2 are special cases of the sufficient conditions for weak lumpability that Serfozo gives in [16]. Theorem 2.2.1 says that if γ is a left eigenvector of $Q(t)$ for every t then $(Q,\gamma) \sim r$ where $r(t)$ is the eigenvalue of $Q(t)$ corresponding to the eigenvector γ. Theorem 2.2.2 is a special case of the necessary and sufficient condition for strong lumpability in Serfozo [16]. Theorem 2.2.2 says that if the row sums of the matrix $Q(t)$ are the same for every t then $Q \approx r$ where $r(t)$ is the common value of the row sums. If the row sums are all the same, the times between transitions do not depend on the state of the process. Thus

$$P(T_n \le t_n | X_{n-1}, X_{n-2}, \cdots X_0, T_{n-1}, \cdots, T_1) = P(T_n \le t_n | X_{n-1}) = P(T_n \le t_n)$$

so $\{T_1, T_2, \cdots, T_n\}$ is a renewal process. The intuitive justification of theorem 2.2.1 is less obvious, but most interesting cases of equivalence seem to be of that type. It will be shown in section four that Burke's theorem is a corollary of theorem 2.2.1.

In everything that follows, the topology on $\mathbb{R}^n$, $n \le \infty$, will be the L_1 topology (i.e. if $x_k \in \mathbb{R}^n$, $K = 1,2,\cdots$ then $x_k \to x$ if $x_k U \to xU$).

Let (Q,γ) be an n state MRP ($n \le \infty$) and let r be the renewal process with distribution $\gamma Q(t)U$. We define the following subsets of $\mathbb{R}^n$.

<u>Definition</u> 2.2.1.

(A) Let $\mathscr{A}_\gamma = \{v \in \mathbb{R}^n\colon vQ(t)U = (vU)r(t),\ \forall t\}$.

(B) Let $\mathscr{V}_\gamma$ be the largest subset of $\mathscr{A}_\gamma$ that is invariant under multiplication by $Q(t)$ (i.e. $\mathscr{V}_\gamma Q(t) \subset \mathscr{V}_\gamma$, $\forall t$).

(C) Let $\mathscr{P} = \{v \in \mathbb{R}^n\colon v \ge 0,\ vU = 1\}$. $\mathscr{P}$ is the set of probability vectors.

(D) Let $\mathscr{K}_\gamma = \mathscr{V}_\gamma \cap \mathscr{P}$.

Lemma 2.2.3. $\mathscr{V}_\gamma$ is well defined.

Proof. Let $\mathscr{A}$ be the class of all subsets of $\mathscr{S}_\gamma$ that are invarient under multiplication by each $Q(t)$. Let $\{A_\alpha\}$, $\alpha \in I$ be an increasing chain in $\mathscr{A}$. Since $\cup A_\alpha$ is invariant and is an upper bound for the chain, there is a maximal element of $\mathscr{A}$ (Zorn's Lemma).□

Lemma 2.2.4. $\mathscr{S}_\gamma$ and $\mathscr{V}_\gamma$ are subspaces of $\mathbb{R}^n$.

Proof. Clearly $\{\gamma, 0\} \subset \mathscr{S}_\gamma$, so $\mathscr{S}_\gamma$ has at least two elements. Let $\beta_1, \beta_2 \in \mathscr{S}_\gamma$. Then $\beta_1 Q(t)U = r(t)(\beta_1 U)$ and $\beta_2 Q(t)U = r(t)(\beta_2 U)$. Thus $(a\beta_1 + b\beta_2)Q(t)U = r(t)(a\beta_1 + b\beta_2)U$, so $a\beta_1 + b\beta_2 \in \mathscr{S}_\gamma$. Thus $\mathscr{S}_\gamma$ is a subspace.

Let $\mathscr{W}$ be any invariant subset of $\mathscr{S}_\gamma$ and let $\mathscr{W}^*$ be the subspace generated by $\mathscr{W}$. Say $w \in \mathscr{W}^*$. $w = \sum_{i=1}^{k} a_i w_i$ where $w_i \in \mathscr{W}$. But $wQ(t) = \sum_{i=1}^{k} a_i w_i Q(t) = \sum_{i=1}^{k} a_i w_i'$ where $w_i' \in \mathscr{W}$ since $\mathscr{W}$ is invariant. Thus $wQ(t) \in \mathscr{W}^*$ so $\mathscr{W}^*$ is invariant. Since $\mathscr{V}_\gamma$ is defined to be the largest invariant set it must be a subspace. □

Lemma 2.2.5. $\mathscr{K}_\gamma$ is closed, convex and invariant under multiplication by $Q(t)/r(t)$, $\forall t$. If $n < \infty$, $\mathscr{K}_\gamma$ is compact.

Proof. Since $\mathscr{V}_\gamma$ is a subspace, $\mathscr{K}_\gamma$ is closed and convex. Let $\beta \in \mathscr{K}_\gamma$. Since $\mathscr{K}_\gamma \subset \mathscr{V}_\gamma$, $\beta Q(t) = r(t)\beta'$ where $\beta' \in \mathscr{V}_\gamma$, and since $\mathscr{V}_\gamma \subset \mathscr{S}_\gamma$, $\beta Q(t)U = (\beta U)r(t) = r(t)$. Thus $\beta' U = 1$ so $\beta' \in \mathscr{K}_\gamma$. Since $\mathscr{K}_\gamma$ is closed and convex it is compact whenever $n < \infty$. □

Corollary 2.2.5.1. If $Q(\infty)$ is irreducible and persistent, the steady state vector $\pi \in \mathscr{K}_\gamma$.

Proof. Say $Q(\infty)$ has period m. Then

$$\lim_{n\to\infty} \frac{1}{m} \sum_{j=n}^{n+m-1} Q^j(\infty) = U\pi.$$

Thus,

$$\forall \beta \in \mathscr{K}_\gamma,\ \beta(\lim_{n\to\infty} \frac{1}{m} \sum_{j=n}^{n+m-1} Q^j(\infty)) = \pi.$$

Since $\mathscr{K}_\gamma$ is closed and convex, $\pi \in \mathscr{K}_\gamma$. □

Consider the column vector $Q(t)U$. If $\beta_1, \beta_2 \in \mathscr{K}_\gamma$ then $\beta_1 Q(t)U = \beta_2 Q(t)U$, $\forall t$ which says $(\beta_1 - \beta_2)$ is orthogonal to $Q(t)U$, $\forall t$.

Lemma 2.2.6. If (Q,γ) is an n state MRP ($n < \infty$) and there exist times $t_1, t_2, \cdots, t_n$ such that $\{Q(t_1)U, Q(t_2)U, \cdots, Q(t_n)U\}$ is a linearly independent set then $\mathscr{S}_\gamma$ is the subspace of $\mathbb{R}^n$ generated by $\{\gamma\}$.

Proof. If $\beta \in \mathscr{S}_\gamma$ then $(\frac{\beta}{\beta U} - \gamma) \perp Q(t)U$, $\forall t$. But only the zero vector can be orthogonal to n independent vectors in $\mathbb{R}^n$. Thus, the only elements of $\mathscr{S}_\gamma$ are the points on the line through γ and the origin. □

We now show the importance of the set $\mathscr{K}_\gamma$.

Theorem 2.2.7. $(Q,\gamma) \sim r$ *if and only if* $\gamma \in \mathscr{K}_\gamma$.

Proof. ($\Longrightarrow$) If $(Q,\gamma) \sim r$ then $\forall n, t_1, t_2, \cdots, t_n$, $\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$. Say there exists $t_1, t_2, \cdots, t_k$ such that $\gamma Q(t_1)Q(t_2)\cdots Q(t_k) \notin \mathscr{S}_\gamma$. Then there is a t such that $\gamma Q(t_1)Q(t_2)\cdots Q(t_k)Q(t)U \neq (\gamma Q(t_1)Q(t_2)\cdots Q(t_k)U)r(t)$. But this says $\gamma Q(t_1)Q(t_2)\cdots Q(t_k)Q(t)U \neq r(t_1)r(t_2)\cdots r(t_k)r(t)$ which contradicts $(Q,\gamma) \sim r$. Thus $\forall k, t_1, t_2, \cdots, t_k$ we have $\gamma Q(t_1)Q(t_2)\cdots Q(t_k) \in \mathscr{S}_\gamma$. Let $\mathscr{W} = \{\gamma\} \cup \{w\colon w = \gamma Q(t_1)Q(t_2)\cdots Q(t_n)$ for some $n, t_1, t_2, \cdots, t_n\}$. $\mathscr{W} \subset \mathscr{S}_\gamma$ and $\mathscr{W}$ is invariant under multiplication by $Q(t)$ so $\mathscr{W} \subset \mathscr{V}_\gamma$. Since $\gamma \in \mathscr{W} \cap \mathscr{P}$ we have $\gamma \in \mathscr{K}_\gamma$.

($\Longleftarrow$) If $\gamma \in \mathscr{K}_\gamma$ then $\forall n, t_1, t_2, \cdots, t_n$, $\gamma Q(t_1)Q(t_2)\cdots Q(t_n) = r(t_1)r(t_2)\cdots r(t_n)\gamma'$ where $\gamma' \in \mathscr{K}_\gamma$. Thus $\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$ so $(Q,\gamma) \sim r$. □

Corollary 2.2.7.1. If Q is an irreducible MRP with steady state vector π, and $r(t) = \pi Q(t)U$ then $Q \sim r$ if and only if $\mathcal{K}_\pi \neq \phi$.

Proof. $(\longrightarrow)$ By Corollary 2.2.5.1, if $(Q,\gamma) \sim r$ for some γ then $(Q,\pi) \sim r$ also. Thus $\pi \in \mathcal{K}_\pi$ by theorem 2.2.7.

$(\longleftarrow)$ If $\mathcal{K}_\pi \neq \phi$ then $\pi \in \mathcal{K}_\pi$ by Corollary 2.2.5.1, so $(Q,\gamma) \sim r$. □

Example 2.2.1. Let $0 < t^* < t^{**}$ and

$$Q(t) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{if } t < t^* \\[2ex] \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} & \text{if } t^* < t < t^{**} \\[2ex] \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} & \text{if } t > t^{**}. \end{cases}$$

$$\gamma = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \quad \pi = (\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}), \quad r(t) = \begin{cases} 0 & \text{if } t < t^* \\ \frac{1}{2} & \text{if } t^* < t < t^{**} \\ 1 & \text{if } t > t^{**}. \end{cases}$$

Claim: $(Q,\pi) \sim r$.

Proof.

$$\pi Q(t) = \begin{cases} r(t)\gamma & \text{if } t < t^{**} \\ r(t)\pi & \text{if } t > t^{**} \end{cases} \qquad \gamma Q(t) = \begin{cases} r(t)\gamma & \text{if } t < t^{**} \\ r(t)\pi & \text{if } t > t^{**}. \end{cases}$$

Thus, $\forall n, t_1, t_2, \cdots t_n$, $\pi Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$. Clearly $(Q,\gamma) \sim r$ also. Note that $Q(t)U \neq r(t)U$ if $t^* < t < t^{**}$, $\pi Q(t) \neq r(t)\pi$ if $t < t^{**}$ and $\gamma Q(t) \neq r(t)\gamma$ if $t > t^{**}$ so the conditions of theorem 2.2.1 and 2.2.2 are not satisfied. In this example $\mathcal{K}_\pi = \mathcal{K}_\gamma \neq \phi$.

Theorem 2.2.1 and 2.2.2 give sufficient conditions for $(Q,\gamma) \sim r$ which are relatively easy to use in practice. Theorem 2.2.7 gives a necessary and sufficient condition, but there is not yet any simple way to determine what $\mathcal{K}_\gamma$ is in general. The following theorems give necessary conditions for $(Q,\gamma) \sim r$ which are also useful in practice. In everything that follows, $\frac{Q(t)}{r(t)} = 0$ if $r(t) = 0$.

Theorem 2.2.8. If (Q,γ) is a finite state MRP and $(Q,\gamma) \sim r$ then $r(t)$ is an eigenvalue of $Q(t)$ for each t. If in addition $Q(t)$ is irreducible for each t, $r(t)$ is the largest eigenvalue of $Q(t)$.

Proof. If (Q,γ) is a finite state MRP, $\mathcal{K}_\gamma$ is compact and convex and satisfies $\mathcal{K}_\gamma Q(t) \subset r(t)\mathcal{K}_\gamma$, $\forall t$. Thus, by the Brouwer fixed point theorem, for each t there is a γ_t such that $\gamma_t Q(t) = r(t)\gamma_t$, so $r(t)$ is an eigenvalue.

Say $Q(t)$ is irreducible. Assume λ is the largest eigenvalue of $Q(t)$ (which must be positive) and $\lambda > r(t)$. Thus the largest eigenvalue of $Q(t)/r(t)$ is strictly greater than one. This implies that $\gamma Q(t)/r(t))^n U$ diverges. But if $(Q,\gamma) \sim r$ then $\gamma(Q(t)/r(t))^n U = 1$ for all n, so $r(t)$ must be the largest eigenvalue.□

Example 2.2.2. If (Q,γ) is an infinite state MRP and $(Q,\gamma) \sim r$ it may be that $r(t)$ is not an eigenvalue of $Q(t)$. For instance, say $0 < t^* < t^{**}$ and

$$
Q(t) = \begin{cases}
\begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix} & \text{if } t < t^{*} \\[2ex]
\begin{bmatrix} \frac{1}{2} & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{2} & & \\ & & & \frac{1}{2} & \\ & & & & \ddots \end{bmatrix} & \text{if } t^{*} \le t < t^{**} \\[2ex]
\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{1}{2} & & \frac{1}{2} & & & \\ \frac{1}{2} & & & \frac{1}{2} & & \\ \frac{1}{2} & & & & \frac{1}{2} & \\ \frac{1}{2} & & & & & \ddots \\ \vdots & & & & & \end{bmatrix} & \text{if } t > t^{**}
\end{cases}
$$

By theorem 2.2.2, $Q \approx r$ where $r(t) = \begin{cases} 0 & \text{if } t < t^{*}, \\ \frac{1}{2} & \text{if } t^{*} \le t < t^{**}, \\ 1 & \text{if } t \ge t^{**}, \end{cases}$

but clearly $r(t)$ is not an eigenvalue of $Q(t)$ for $t^{*} \le t < t^{**}$.

Although it is unrealistic to expect to be able to check to see whether $r(t)$ is an eigenvalue of $Q(t)$ for each t, theorem 2.2.8 says that one can show that $Q \not\to r$ by merely finding a value of t where $r(t)$ is

not an eigenvalue. The following theorem is useful in the same way.

Theorem 2.2.9. If (Q,γ) is an n state MRP $(n < \infty)$ and there exists times $t_1,t_2,\cdots,t_n$ such that $\{Q(t_1)U,Q(t_2)U,\cdots,Q(t_n)U\}$ is a linearly independent set then $(Q,\gamma) \sim r$ if and only if $\forall t,\ \gamma Q(t) = r(t)\gamma$.

Proof. By lemma 2.2.6, $\mathcal{S}_\gamma$ is a one dimensional subspace so either $\mathcal{V}_\gamma = \mathcal{S}_\gamma$ or $\mathcal{V}_\gamma = \{0\}$. By theorem 2.2.7, $(Q,\gamma) \sim r$ implies $\gamma \in \mathcal{K}_\gamma$, so $\mathcal{K}_\gamma$ must consist of the single vector $\{\gamma\}$. Since $\mathcal{K}_\gamma$ is invariant under multiplication by $Q(t)/r(t)$, we have $\gamma Q(t) = r(t)\gamma,\ \forall t$. The converse is a restatement of theorem 2.2.1. □

Corollary 2.2.9.1. If Q is an n state irreducible MRP $(n < \infty)$ with steady state vector π and there exists times $t_1,t_2,\cdots,t_n$ such that $\{Q(t_1)U,Q(t_2)U,\cdots,Q(t_n)U\}$ is a linearly independent set then $Q \sim r$ if and only if $(Q,\pi) \sim r$.

Proof. If Q is irreducible and $(Q,\gamma) \sim r$ then $(Q,\pi) \sim r$ also. But the theorem says that at most one initial distribution can yield a renewal process. Thus either $(Q,\pi) \sim r$ or $Q \not\sim r$. □

3. A Rough Algorithm for Determining Whether $Q \sim r$.

In general, a MRP (Q,γ) is not equivalent to the renewal process γQU. In fact, for a given MRP, Q, there is usually no initial distribution that yields a renewal process. In many queueing and other processes there are random processes that can be easily characterized as MRP's, but the interesting and important question is whether or not the random process is a renewal process (see the examples in the next section).

Unless there is a very good reason to believe that the process is renewal it makes sense to first try to show that there is no initial distribution that yields a renewal process. If Q is irreducible (as it

usually will be in practice) the simplest and most successful approach is to use corollary 2.2.5.1 and the fact that if $\gamma_1, \gamma_2 \in \mathcal{K}_\gamma$ then $(\gamma_1 - \gamma_2) \perp Q(t)U$, $\forall t$ is a necessary condition for $(Q,\gamma) \sim r$. Corollary 2.2.5.1 says that if Q is irreducible with steady state vector π then $\pi \in \mathcal{K}_\gamma$ whenever $\mathcal{K}_\gamma \neq \phi$. Since $\mathcal{K}_\gamma$ is invariant under multiplication by $Q(t)/r(t)$ we must have $(\pi - \frac{\pi Q(t)}{r(t)})Q(x)U = 0$, $\forall t, x$. If $Q \nsim r$ there is often a simple argument showing one will not get zero for all values of t and x (e.g. example 2.4.3 in the next section). Otherwise (as in example 2.4.4) the multiplication has to be carried out. Of course it is possible that $(\pi - \frac{\pi Q(t)}{r(t)})Q(x)U = 0$, $\forall t$, x even if $(Q,\pi) \nsim r$ since $\pi Q(t)Q(x)U = r(t)r(x)$, $\forall t$, x is not a sufficient condition for $(Q,\pi) \sim r$. An example where this occurs would be very difficult to construct, though, and it is very doubtful that one would ever come across one in practice. If this multiplication appears to be difficult it may also be possible to show that $Q \nsim r$ by showing that r(t) is not an eigenvalue of Q(t) for some t. (see example 4.2.1)

If the tests to show that $Q \nsim r$ fail, there is every reason to believe that $Q \sim r$. To verify that $Q \sim r$ it makes sense to show that $\pi Q(t) = r(t)\pi$, $\forall t$ where π is the steady state vector or that $Q(t)U = Ur(t)$, $\forall t$ (theorems 2.2.1, 2.2.2). If Q has n states $(n < \infty)$ and $\{Q(t_j)U\}$ $j = 1,2,\cdots,n$ is a linearly independent set for some $t_1, t_2, \cdots, t_n$, theorem 2.2.9 says that $\mathcal{K}_\gamma = \phi$ if $\gamma \neq \pi$ and $\pi Q(t) = r(t)\pi$, $\forall t$ becomes a necessary (and sufficient) condition for $Q \sim r$. (see example 2.4.1).

4. Examples and Applications

Example 2.4.1. Disney, Farrell and DeMorais [4] show that the output of an M/D/1/1 queue in steady state is a renewal process. The

results obtained thus far allow for a quick verification of that fact.

The output from an M/D/1/1 queue is a two state MRP with kernel

$$Q(t) = \begin{pmatrix} Q_{00}(t) & Q_{01}(t) \\ Q_{10}(t) & Q_{11}(t) \end{pmatrix},$$

where $Q_{ij}(t)$ is the probability that given a customer departs (at time zero) leaving i customers in the queue, the next departure occurs before time t, and when that customer leaves there are j customers left in the queue.

If arrivals are Poisson with rate λ and the service times are deterministic with duration d then

$$Q(t) = \begin{pmatrix} e^{-\lambda d}(1-e^{-\lambda(t-d)}) & (1-e^{-\lambda d})(1-e^{-\lambda(t-d)}) \\ e^{-\lambda d} & 1-e^{-\lambda d} \end{pmatrix} 1_d(t),$$

where $1_d(t) = \begin{cases} 0 & \text{if } t < d \\ 1 & \text{if } t \geq d. \end{cases}$

The embedded Markov chain has transition probability matrix

$$Q(\infty) = \begin{pmatrix} e^{-\lambda d} & 1-e^{-\lambda d} \\ e^{-\lambda d} & 1-e^{-\lambda d} \end{pmatrix},$$

so the steady state vector is $\pi = (e^{-\lambda d}, 1-e^{-\lambda d})$. Performing the multiplication $\pi Q(t)$ we get

$$\pi Q(t) = ((1-e^{-\lambda t})e^{-\lambda d}, \ (1-e^{-\lambda t})(1-e^{-\lambda d}))1_d(t) = (1-e^{-\lambda t})1_d(t)\pi.$$

By theorem 2.2.1 we know $(Q,\pi) \sim r$ where $r(t) = (1-e^{-\lambda t})1_d(t)$.

It is easily verified that for any $t_1 \neq t_2$ the vectors $Q(t_1)U$ and $Q(t_2)U$ are linearly independent. Thus, by theorem 2.2.9, $(Q,\gamma) \sim r$ if and only if $\gamma Q(t) = r(t)\gamma$ for each t. By the Perron-Frobenius theorem we must have $\gamma = \pi$. Thus π is the only initial distribution that yields a renewal process.

Example 2.4.2. Burke's theorem [2] implies that the output from a steady state M/M/1 queue is a Poisson process. The output process is a MRP with kernel $Q(t)$ where $Q_{ij}(t)$ has the same interpretation as in the first example except that in this case i and j range over all the non-negative integers. If the arrival rate is λ and the service rate is μ, the kernel has the form

$$Q(t) = \begin{pmatrix} q_0(t) & q_1(t) & q_2(t) & \cdots \\ f_0(t) & f_1(t) & f_2(t) & \cdots \\ & f_0(t) & f_1(t) & \cdots \\ & & f_0(t) & \cdots \\ & & & \ddots \end{pmatrix},$$

where

$$f_j(t) = \int_0^t \frac{(\lambda s)^j}{j!} e^{-\lambda s} \mu e^{-\mu s} ds \quad \text{and}$$

$$q_j(t) = \int_0^t \lambda e^{-\lambda s} f_j(t-s) ds.$$

It is well known that the steady state vector for this MRP is $\pi = (1-\rho)(1,\rho,\rho^2,\cdots)$, where $\rho = \lambda/\mu$. Performing the multiplication $\pi Q(t)$ we get

$$(\pi Q(t))_j = (1-\rho)^{-1}(q_j(t) + \sum_{k=0}^{j} f_{j-k}(t)\rho^{k+1})$$
$$= (1-e^{-\lambda t})(1-\rho)^{-1}\rho^j = (1-e^{-\lambda t})\pi_j.$$

Thus, the steady state output is a renewal process with distribution $1-e^{-\lambda t}$ (a Poisson process).

Example 2.4.3. Now consider the M/M/1/N queue ($N < \infty$). The output from this queue is an $N + 1$ state MRP with kernel

$$Q_N(t) = \begin{bmatrix} q_0(t) & q_1(t) & q_2(t) & \cdots & q_{N-1}(t) & \sum_{j=N}^{\infty} q_j(t) \\ f_0(t) & f_1(t) & f_2(t) & \cdots & f_{N-1}(t) & \sum_{j=N}^{\infty} f_j(t) \\ & f_0(t) & f_1(t) & \cdots & f_{N-2}(t) & \sum_{j=N-1}^{\infty} f_j(t) \\ & & f_0(t) & \cdots & f_{N-3}(t) & \sum_{j=N-2}^{\infty} f_j(t) \\ & & & \ddots & \vdots & \vdots \\ & & & & f_0(t) & \sum_{j=1}^{\infty} f_j(t) \end{bmatrix}$$

where $q_j(t)$ and $f_j(t)$ are as in the last example. The steady state vector is $\pi^N = \frac{1-\rho}{1-\rho^{N+1}} (1,\rho,\cdots,\rho^N)$.

If the steady state output from the M/M/1/N queue is a renewal process then the renewal process would have to have distribution

$$r_N(t) = \pi^N Q_N(t)U = 1 - \frac{e^{-\lambda t}}{1-\rho^{N+1}} + \frac{\rho^{N+1}e^{-\mu t}}{1-\rho^{N+1}} .$$

If $\pi^N \in \mathscr{K}_\gamma$ (for some γ) then so is $\frac{\pi^N Q_N(x)}{r_N(x)}$ for any x. Thus,

$$\forall t, x, \left(\pi^N - \frac{\pi^N Q_N(x)}{r_N(x)}\right) \perp Q(t)U.$$

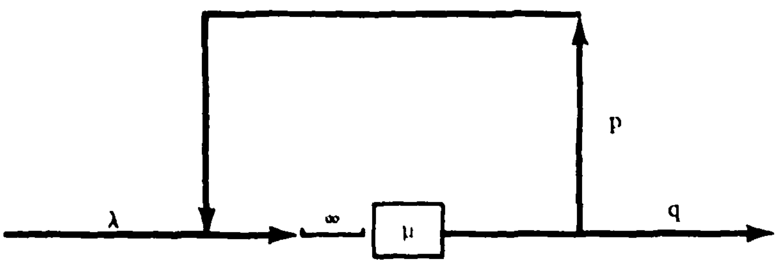
Clearly $Q_N(t)U$ has the form $(a,b,b,\cdots,b)$ and $r_N(x) < 1-e^{-\lambda x}$.

Also, since $\pi^N Q_N(x) = \frac{1-\rho}{1-\rho^{N+1}}(1-e^{-\lambda x})(1,\rho,\cdots,\rho^{N-1},K(x))$ where

$K(x) = \frac{r_N(x)}{1-e^{-\lambda x}} \frac{1-\rho^{N+1}}{1-\rho} - \frac{1-\rho^N}{1-\rho}$, a straight forward calculation yields

$\left(\pi^N - \frac{\pi^N Q_N(x)}{r_N(x)}\right) Q(t)U \neq 0$ for any t, $x > 0$. By corollary 2.2.5.1, the output from an M/M/1/N queue is not a renewal process for any initial distribution.

<u>Example</u> 2.4.4. Consider the following Jackson network.



There is a Poisson arrival stream with rate λ, an exponential server with rate μ, and a Bernoulli switch that feeds a departing customer back to the end of the line with probability p. The input process is defined to be the superposition of the arrival process and the feedback process. (see [5] for a complete discussion of this problem). The input process is a MRP with kernel $Q(t)$ where

$$Q_{ij}(t) = \begin{cases} 0 \quad \text{if } j > i+1 \\ \int_0^t (e^{-\lambda s} - qe^{-\lambda t}) q^i \dfrac{\mu(\mu s)^i e^{-\mu s}}{i!} \, ds \quad \text{if } j = 0,\ i \geq 0, \\ \int_0^t e^{-\lambda s} (\dfrac{q\lambda}{\lambda+\mu} (1 - e^{-(\lambda+\mu)(t-s)}) + p) q^{i-j} \dfrac{\mu(\mu s)^{i-j} e^{-\mu s}}{(i-j)!} \quad \text{if } 1 \leq j \leq i, \\ e^{-\mu t}(1 - e^{-\lambda t}) \quad \text{if } j = i+1. \end{cases}$$

Since the system is a Jackson network, the steady state queue length probabilities embedded at inputs is known to be

$$\pi = (1 - \frac{\lambda}{\mu q})(1, \frac{\lambda}{\mu q}, (\frac{\lambda}{\mu q})^2, \cdots).$$

If the input process were renewal it would have distribution

$$r(t) = \pi Q(t)U = 1 - \frac{q\mu - \lambda}{\mu - \lambda} e^{-\lambda t} - \frac{p\mu}{\mu - \lambda} e^{-\mu t},$$

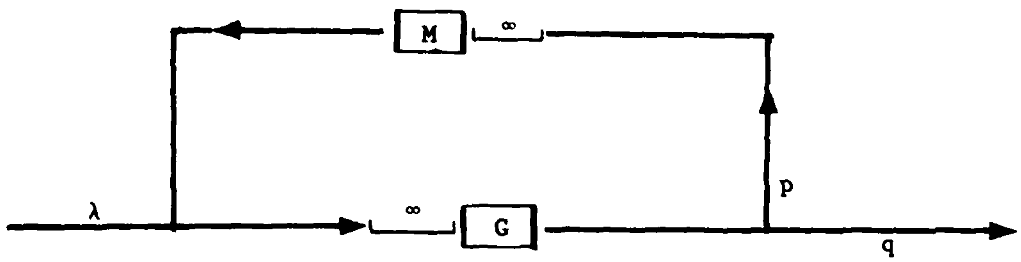
and $\forall x,y,\ (\pi - \frac{\pi Q(x)}{r(x)}) Q(y)U = 0$. But

$$(\pi - \frac{\pi Q(x)}{r(x)}) Q(y)U = \frac{r(x) - (1 - e^{-\mu x})}{r(x)} (\frac{p\lambda}{\mu - \lambda} e^{-\lambda y} - \frac{p\mu}{\mu - \lambda} e^{-\mu y} + pe^{-(\mu+\lambda)y}).$$

Since $r(x) - (1 - e^{-\mu x}) < 0$, the process cannot be renewal if for some y $\frac{p\lambda}{\mu - \lambda} e^{-\lambda y} - \frac{p\mu}{\mu - \lambda} e^{-\mu y} + pe^{-(\mu+\lambda)y} \neq 0$. By the linear independence of exponentials, this expression can only be zero if $p = 0$. Thus if there is a positive probability of a customer feeding back, the input process is not renewal. If $p = 0$, a direct calculation shows that $Q(t)U = (1 - e^{-\lambda t})U$ (as it obviously should be). By theorem 2.2.2 the input process is a Poisson process in that case. The question of whether or not the input process of this network is renewal was previously

unresolved.

<u>Example</u> 2.4.5. Now consider the M/G/1 queue with delayed feedback, where the delay mechanism is a $\cdot$/M/1 queue.



A full discussion of this system can be found in Foley [7]. It is shown that if the departure process is renewal it must be Poisson with rate λ. The idea of equivalence yields an interesting corollary to that fact.

Assume the G server has service time distribution $g(t)$ where g is differentiable at zero and $g'(0) = \mu$. The departure process is most naturally represented by a MRP with state space $S = \{0,1,2,\cdots\} \times \{0,1,2,\cdots\}$ since the future of the system is conditionally independent of the past at a departure point given the length of both queues.

Let $\gamma = \{\gamma_{ij}: i,j = 0,1,2,\cdots$, lexicographically ordered$\}$ be a distribution on S. Let $Q(t)$ be the kernel of the MRP representing the departure process.

Clearly $Q(t)$ has the form

$$\begin{array}{|c|c|c|c|c|}
\hline
Q_{00}(t) & Q_{01}(t) & Q_{02}(t) & Q_{03}(t) & \cdots \\
\hline
Q_{10}(t) & Q_{11}(t) & Q_{12}(t) & Q_{13}(t) & \cdots \\
\hline
Q_{20}(t) & Q_{21}(t) & Q_{22}(t) & Q_{23}(t) & \cdots \\
\hline
Q_{30}(t) & Q_{31}(t) & Q_{32}(t) & Q_{33}(t) & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
\end{array}$$

where $Q_{ij}(t)$ is the matrix $Q_{(i,i')(j,j')}(t)$, $i',j' \in \{0,1,2,\cdots\}$.

Although $Q(t)$ cannot be easily written down, it is a simple matter to verify that

$$Q'(0) = \begin{array}{|c|c|c|c|c|}
\hline
 & & & & \\
\hline
\mu q I & & & & \\
\hline
 & \mu q I & & & \\
\hline
 & & \mu q I & & \\
\hline
 & & & \ddots & \\
\hline
\end{array} \; .$$

Say $(Q,\gamma) \sim r$. We know that $r(t) = 1 - e^{-\lambda t}$, so by taking derivatives the definition of equivalence yields

$$\forall n, \quad \gamma Q'(0)^n U = \lambda^n .$$

To justify the last step note that $Q'(0) = \lim_{t\to 0} \frac{Q(t)}{t}$ since $Q(0) = 0$. Thus

$$\gamma Q'(0) = \lim_{t\to 0} \frac{\gamma Q(t)}{t} = \lim_{t\to 0} \frac{r(t)}{t} \hat{\gamma}_t$$

where $\hat{\gamma}_t \in \mathscr{K}_\gamma$. Likewise

$$\gamma Q'(0)^n = \lim_{t_n\to 0} \lim_{t_{n-1}\to 0} \cdots \lim_{t_1\to 0} \frac{r(t_1)r(t_2)\cdots r(t_n)}{t_1 t_2 \cdots t_n} \hat{\gamma}_{t_1 t_2 \cdots t_n}$$

where $\hat{\gamma}_{t_1 t_2 \cdots t_n} \in \mathscr{K}_\gamma$. Since $\mathscr{K}_\gamma$ is closed,

$$\gamma Q'(0)^n = r'(0)^n \hat{\gamma}$$

where $\hat{\gamma} \in \mathscr{K}_\gamma$, so $\gamma Q'(0)^n U = r'(0)^n$.

Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \cdots)$ where $\gamma_j = (\gamma_{j0}, \gamma_{j1}, \gamma_{j2}, \cdots)$. By the form of $Q'(0)$ we have

$$\gamma Q'(0)^n = (\overbrace{0, 0, \cdots, 0,}^{n} (\mu q)^n \gamma_n, (\mu q)^n \gamma_{n+1}, \cdots)$$

so $\sum_{i=n}^{\infty} \gamma_i U = (\frac{\lambda}{\mu q})^n$. This implies $\gamma_n U = (\frac{\lambda}{\mu q})^n (1 - \frac{\lambda}{\mu q})$. The conclusion is that if the departure process is renewal, the marginal queue length distribution at the G server must be geometric with parameter $\frac{\lambda}{\mu q}$.

Say the G server has a distribution $g(t)$ that is not differentiable at zero. If the departure process is renewal we must have

$$\frac{\gamma(Q(t) - Q(0))U}{t} = \frac{r(t) - r(0)}{t}$$

for all $t > 0$, and since $r(t) = 1 - e^{-\lambda t}$ we must have

$$\lim_{t\to 0} \frac{\gamma(Q(t) - Q(0))U}{t} = \lambda.$$

If $g(t)$ is not right-differentiable at zero the limit does not exist so

the departure process cannot be renewal.

Example 2.4.6. Consider an M/M/1/N queue ($N > 0$) where the arrival rate and service rate depend on the length of the queue. Let $\lambda_n > 0$ and $\mu_n > 0$ be the arrival and service rate when the queue length is n. Assume $\limsup \frac{\lambda_{n+1}}{\mu_n} < 1$ to assure a steady state. Let $Q(t)$ be the kernel of the MRP associated with the departure process. If $N < \infty$, Q has $N + 1$ states and

$$Q'(0) = \begin{bmatrix} & & & & \\ \mu_1 & & & & \\ & \mu_2 & & & \\ & & \ddots & & \\ & & & \mu_N & \end{bmatrix}$$

Assume $(Q,\pi) \sim r$ for some r, where π is the steady state vector. $\pi Q'(0)U = r'(0)$ so $r'(0) = \sum_{j=1}^{N} \pi_j \mu_j$. But, for $n > N$, $Q'(0)^n = 0$ so $0 = \pi Q'(0)^n U = r'(0)^n$. Thus, $r'(0) = 0$, which implies $\pi_j = 0$, $j = 1,2,\cdots,N$. Since Q is irreducible, this is impossible so we conclude that the departure process can never be renewal.

If $N = \infty$ the situation changes slightly since $Q'(0)^n$ is never zero. First consider the case where $\mu_j = \mu$, $j = 1,2,\cdots$. In this case $(Q,\gamma) \sim r$ implies $\gamma Q'(0)^n U = r'(0)^n$, $\forall n$, which implies

$$\gamma_n = \left(\frac{r'(0)}{\mu}\right)^n \left(1 - \frac{r'(0)}{\mu}\right).$$

It is easily verified that the steady state queue length distribution is π, where

$$\pi_j = \frac{\lambda_0\lambda_1\cdots\lambda_j}{\mu^j}\,\pi_0.$$

This is not geometric unless $\lambda_j = \lambda$, $j = 0,1,2,\cdots$. Thus the departure process cannot be renewal unless λ_j is a constant (i.e. the M/M/1 queue).

If we allow μ_j to depend on j the equations $\gamma Q'(0)^n U = r'(0)^n$ yield

$$\gamma_j = \frac{r'(0)^n}{\mu_1\mu_2\cdots\mu_j}\,\gamma_0.$$

Since the steady state queue length distribution is

$$\pi_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\cdots\mu_j}\,\pi_0,$$

it is again impossible to have a renewal departure process unless λ_j is a constant (i.e. $\lambda_j = r'(0)$).

We cannot deduce from these equations that μ_j must also be constant. In fact, by [13, theorem 3], any birth-death queue with a steady state and $\lambda_n = \lambda$ has a Poisson departure process with rate λ. For example, let $\lambda_n = \lambda$ and say

$$\mu_j = \begin{cases} j\mu, & j = 0,1,2,\cdots,N \\ N\mu, & j > N. \end{cases}$$

This is precisely the M/M/N queue, which is known to have a Poisson departure process with rate λ when in steady state.

If we relax our assumption that the μ_n's and λ_n's be nonzero there are other classes of queues with renewal departures. If $\mu_n = 0$ the MRP is no longer irreducible since the queue length can never be less than n once it is greater than n. Likewise, if $\lambda_n = 0$ the queue length

can never exceed n once it is below n. An invariant queue length distribution need not be strictly positive.

Let $N < \infty$. Say $(Q,\gamma) \sim r$ for some γ and r. Let $\pi = \lim_{n\to\infty} \gamma Q^n(\infty)$ be an invariant distribution that satisfies $(Q,\pi) \sim r$. Say $\pi_0 > 0$. If $\lambda_0 > 0$ we have $\pi_1 > 0$ which implies $\mu_1 = 0$ which implies $\pi_0 = 0$; a contradiction. If $\lambda_0 = 0$ either $\pi_0 = 1$, in which case there is no departure process, or $\pi_0 = 0$.

Say $\pi_j > 0$ for some $j > 0$. Then $\mu_j = 0$, so $\pi_k = 0$, $k = 0,1,2,\cdots,j-1$. If $\lambda_j = 0$ there is no departure process, so $\lambda_j > 0$. If we also had $\lambda_{j+1} > 0$ we would have $\pi_k > 0$ for some $k > j$ which implies $\pi_j = 0$; a contradication.

The only remaining possibility is that for some j, $\pi = (0,0,\cdots 1,\cdots 0)$, $\mu_j = 0$, $\mu_{j+1} > 0$, $\lambda_j > 0$, $\lambda_{j+1} = 0$ (all other μ's and λ's arbitrary). In this case the queue length is always j after a departure, and the dpearture process is renewal with distribution

$$r(t) = \int_0^t \lambda_j e^{-\lambda_j s}(1 - e^{-\mu_{j+1}(t-s)})ds.$$

Say $N = \infty$ and suppose $(Q,\pi) \sim r$ where π is an invariant distribution. Again, if $\mu_n = 0$ and $\pi_n > 0$ then $\pi_k = 0$, $k < n$. Likewise if $\lambda_m = 0$ and $\lambda_{m+1} > 0$ then $\pi_k = 0$, $k > m + 1$. The arguments given for $N < \infty$ show that the departure process cannot be renewal if $\lambda_m = 0$, $\pi_m > 0$, $\mu_n = 0$, $\pi_n > 0$ unless $n = m + 1$.

Thus, the only other way there can be a renewal departure process is as follows. Say $\mu_n = 0$, $\mu_{n+k} > 0$, $\lambda_{n+k-1} > 0$, $k = 1,2,\cdots$. In this case

$$Q(t) = \begin{array}{r} n\left\{\right. \\ \\ \end{array}\left[\begin{array}{c|c} A(t) & B(t) \\ \hline 0 & \hat{Q}(t) \end{array}\right]$$

where $\hat{Q}$ is irreducible. In fact $\hat{Q}(t)$ is the kernel of the MRP for the departure process from an M/M/1 queue with $\hat{\mu}_k = \mu_{n+k}$, $\hat{\lambda}_{k-1} = \lambda_{n+k-1}$, $k = 1,2,\cdots$. Let π be the steady state vector for $\hat{Q}$. If $(\hat{Q},\hat{\pi}) \sim r$ then $(Q,\pi) \sim r$ where $\pi = (0,\hat{\pi})$.

5. Equivalence as a Homomorphism.

Let (Q,γ) be a MRP and let $r(t) = \gamma Q(t)U$. Associated with each $t \in [0,\infty]$ is a matrix $Q(t)$. Let $\mathcal{Q}$ be the $\mathbb{R}$-algebra of matrices generated by $\{Q(t)\}$, $t \in [0,\infty]$. For each probability vector β we have a map $F_\beta\colon \mathcal{Q} \to \mathbb{R}$ where

$$F_\beta(A) = \beta AU.$$

A necessary and sufficient condition for equivalence can now be written in a very simple form.

Theorem 2.5.1. $(Q,\gamma) \sim r$ if and only if F_γ is a homomorphism.

Proof. ($\Longrightarrow$) If $(Q,\gamma) \sim r$ then

$$\forall n, t_1, t_2, \cdots t_n, \quad \gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n).$$

Also, if A_1, $A_2 \in \mathcal{Q}$ then $\gamma(A_1 + A_2)U = \gamma A_1 U + \gamma A_2 U$ os F_γ is a homomorphism.

($\Longleftarrow$) If F_γ is a homomorphism then $\forall n, t_1, t_2, \cdots t_n$,

$$\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = (\gamma Q(t_1)U)(\gamma Q(t_2)U)\cdots(\gamma Q(t_n)U) = r(t_1)r(t_2)\cdots r(t_n),$$

so $(Q,\gamma) \sim r$. □

The maps $\{F_\beta\}$ also give an alternate characterization of the set $\mathcal{K}_\pi$ when Q is an irreducible MRP with steady state vector π.

Theorem 2.5.2. $\beta \in \mathcal{K}_\pi$ if and only if F_β is a homomorphism.

Proof. $(\Longrightarrow)$ If $\beta \in \mathcal{K}_\pi$ then $(Q,\beta) \sim r$ where $r(t) = \beta Q(t)U$ so F_β is a homomorphism by theorem 2.5.1.

$(\Longleftarrow)$ Let $\mathcal{L} = \{\gamma\colon \gamma = \dfrac{\beta Q(x_1)Q(x_2)\cdots Q(x_m)}{r(x_1)r(x_2)\cdots r(x_m)}$ for some $x_1,x_2,\cdots,x_m\}$.

If $\gamma \in \mathcal{L}$ then F_γ is a homomorphism since

$$\begin{aligned}\gamma Q(t_1)Q(t_2)\cdots Q(t_n)U &= \frac{\beta Q(x_1)Q(x_2)\cdots Q(x_m)}{r(x_1)r(x_2)\cdots r(x_m)}\, Q(t_1)Q(t_2)\cdots Q(t_n)U \\ &= \frac{r(x_1)r(x_2)\cdots r(x_m)r(t_1)r(t_2)\cdots r(t_n)}{r(x_1)r(x_2)\cdots r(x_m)} \\ &= r(t_1)r(t_2)\cdots r(t_n).\end{aligned}$$

Furthermore, $F_\gamma = F_\beta$. Clearly if $\gamma \in \bar{\mathcal{L}}$, $F_\gamma = F_\beta$ and if γ is any linear combination of elements in $\bar{\mathcal{L}}$ then $F_\gamma = F_\beta$. Thus $F_\pi = F_\beta$, so $\beta \in \mathcal{K}_\pi$. □

6. Collapsibility.

Let (Q,π) be a k state MRP $(k \le \infty)$ and let $\{A_1,A_2,\cdots,A_m\}$, $(m < k)$ be a partition of the states of (Q,π). Let Π, U be as defined by (1.2.1) and (1.2.2).

$\Pi Q(t)\underset{\sim}{U}$ is an m x m matrix whose (i,j) element is $P_\pi(X_1 \in A_j,\ T_1 \le t \mid X_0 \in A_i)$. Say F is the function that maps the state space of (Q,π) to $\{A_1,A_2,\cdots,A_m\}$. Serfozo [16] shows that if $\{F(X_n),T_n\}$ is a MRP, its kernel is $\Pi Q(t)\underset{\sim}{U}$.

The definition of equivalence between a MRP and a renewal process has a natural generalization to collapsibility between two MRP's.

<u>Definition</u> 2.6.1. Let (Q,π) be a MRP with state space $\{1,2,\cdots,k\}$ $(k \leq \infty)$ and let Y be an m state MRP $(m \leq k)$. Let $F: \{1,2,\cdots,k\} \to \{A_1,A_2,\cdots,A_m\}$ be a partition of the states of (Q,π). We say (Q,π) is collapsible to Y via the partition F, written $(Q,\pi) \overset{F}{\sim} Y$, if

$$\forall n,t_1,t_2,\cdots,t_n, \quad \Pi Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U} = Y(t_1)Y(t_2)\cdots Y(t_n).$$

If F is clear from the context we will write $(Q,\pi) \sim Y$. Note that the initial distribution of Y depends only on π and F; namely $\gamma = \pi\underset{\sim}{U}$.

<u>Lemma</u> 2.6.1. If $\pi Q(\infty) = \pi$ and $(Q,\pi) \overset{F}{\sim} Y$ then $\gamma Y(\infty) = \gamma$ where $\gamma = \pi\underset{\sim}{U}$.

<u>Proof</u>. Since

$$\Pi_{ij} = \begin{cases} 0 & \text{if } j \notin F^{-1}(i) \\ \dfrac{\pi_j}{\sum\limits_{k\in F^{-1}(i)} \pi_k} & \text{if } j \in F^{-1}(i), \end{cases}$$

we have $\gamma Y(\infty) = \gamma\Pi Q(\infty)\underset{\sim}{U} = \pi\underset{\sim}{U}\Pi Q(\infty)\underset{\sim}{U} = \pi Q(\infty)\underset{\sim}{U} = \pi\underset{\sim}{U} = \gamma.\square$

In the case where Y has one state (a renewal process) we showed that collapsibility is the same as weak lumpability. If (Q,π) is $\{X_n, T_n\}$ and (Y,γ) is $\{Z_n, S_n\}$ (where $\gamma = \pi\underset{\sim}{U}$), then definition 2.6.1 says that for each i, $j \in \{1,2,\cdots,m\}$ and $\forall n,t_1,t_2,\cdots,t_n$,

(2.6.1) $P_\pi(X_n\epsilon A_j,T_n \leq t_n,\cdots,T_1 \leq t_1|X_0\epsilon A_i) = P(Z_n=j,S_n \leq t_n,\cdots,S_1 \leq t_1|Z_0=i)$.

For weak lumpability between $\{X_n, T_n\}$ and $\{Z_n, S_n\}$ we would need that for each i, $j \in \{1,2,\cdots,m\}$ and $\forall n,t_1,t_2,\cdots,t_n$, and $i_1,i_2,\cdots,i_{n-1} \in \{1,2,\cdots,m\}$,

$$(2.6.2)\quad P_{\pi}(X_n \varepsilon A_j, X_{n-1} \varepsilon A_{i_{n-1}}, \cdots, X_1 \varepsilon A_{i_1},\ T_n \le t_n, \cdots, T_1 \le t_1 \mid X_0 \varepsilon A_i)$$

$$= P(Z_n = j, Z_{n-1} = i_{n-1}, \cdots, Z_1 = i_1,\ S_n \le t_n, \cdots, S_1 \le t_1 \mid Z_0 = i).$$

It seems inconceivable that every $\{X_n, T_n\}$ and $\{Z_n, S_n\}$ that satisfy (2.6.1) would also satisfy (2.6.2) but all attempts to find a counterexample have failed so far. Clearly weak lumpability implies collapsibility, though.

Since the definition of collapsibility between two MRP's is analogous to the definition of equivalence between a MRP and a renewal process one might suspect that the conditions for collapsibility would be similar. First of all, any sufficient condition for weak lumpability will also be a sufficient condition for collapsibility. Thus we have

<u>Theorem</u> 2.6.2. If $\Pi Q(t) = Y(t)\Pi$, $\forall t$ then $(Q,\gamma) \overset{F}{\sim} Y$ for any γ that satisfies $\dfrac{\gamma_k}{\sum_{j \varepsilon A_i} \gamma_j} = \Pi_{ik}$ whenever $k \varepsilon A_i$.

<u>Theorem</u> 2.6.3. If $Q(t)\underset{\sim}{U} = \underset{\sim}{U}Y(t)$, $\forall t$ then $(Q,\gamma) \overset{F}{\sim} Y$ for any γ.

Theorems 2.6.2 and 2.6.3 can be proved the same way theorems 2.2.1 and 2.2.2 were proved. They can also be found in Serfozo [15].

Let (Q,Π) be a k state MRP, let Y be an m state MRP and let F be a map from the state space of (Q,π) to the state space of Y. Let $\mathcal{M}_F$ be the set of all m x k matrices, M, with $M_{ij} \ge 0$ and $M\underset{\sim}{U} = \alpha I$ where $\underset{\sim}{U}$ is the summing matrix induced by F and α is a scalar. In other words, M must have the form

$$M = \begin{bmatrix} X\cdots X & & & \\ & X\cdots X & & \\ & & \ddots & \\ & & & X\cdots X \end{bmatrix},$$

and each row sum must be the same. Let $\mathcal{S}_F = \{M \in \mathcal{M}_F : MQ(t)\underset{\sim}{U} = (M\underset{\sim}{U})Y(t), \forall t\}$ and let $\mathcal{V}_F$ be the largest subset of $\mathcal{S}_F$ that satisfies $\mathcal{V}_F Q(t) \subset Y(t)\ \mathcal{V}_F, \forall t$. Let $\mathcal{K}_F = \{M \in \mathcal{V}_F : M\underset{\sim}{U} = 1\}$. These sets are analogous to the sets defined in section 2 and not too surprisingly we have

<u>Theorem</u> 2.6.4. $(Q,\pi) \overset{F}{\sim} Y$ if and only if $\Pi \in \mathcal{K}_F$.

<u>Proof</u>. ($\Longrightarrow$) If $\Pi \notin \mathcal{K}_F$ then $\Pi \notin \mathcal{V}_F$. Thus, for some $t_1, t_2, \cdots, t_n$, $\Pi Q(t_1)Q(t_2)\cdots Q(t_n) \notin \mathcal{S}_F$. But this says that for some t

$$\Pi Q(t_1)Q(t_2)\cdots Q(t_n)Q(t)\underset{\sim}{U} \neq \Pi Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U}Y(t)$$

$$= Y(t_1)Y(t_2)\cdots Y(t_n)Y(t).$$

This contradicts $(Q,\pi) \overset{F}{\sim} Y$.

($\Longleftarrow$) If $\Pi \in \mathcal{K}_F$ then $\forall n, t_1, t_2, \cdots, t_n$,

$$\Pi Q(t_1)Q(t_2)\cdots Q(t_n) \in Y(t_1)Y(t_2)\cdots Y(t_n)\ \mathcal{K}_F$$

so

$$\Pi Q(t_1)Q(t_2)\cdots Q(t_n)U = Y(t_1)Y(t_2)\cdots Y(t_n). \square$$

Another result that carries over easily from the renewal case is that collapsibility is identical to a certain algebra homomorphism. Let $\mathcal{Q}$ be the ring generated by $\{Q(t)\}$, $t \in [0,\infty]$ and let $\mathcal{Y}$ be the ring generated by $\{Y(t)\}$, $t \in [0,\infty]$. Let $\Phi_{\pi,F}: \mathcal{Q} \to \mathcal{Y}$ be $\Phi_{\pi,F}(A) = \Pi A\underset{\sim}{U}$.

<u>Theorem</u> 2.6.5. $(Q,\pi) \overset{F}{\sim} Y$ if and only if $\Phi_{\pi,F}$ is a homomorphism.

The proof here is identical to the proof in the renewal case.

One result that has no counterpart in the renewal case is

<u>Theorem</u> 2.6.6. Let (Q,π), Y and D be k, m and ℓ state MRP's $(k \geq m \geq \ell)$. If $(Q,\pi) \overset{F}{\sim} Y$ and $(Y,\gamma) \overset{G}{\sim} D$ (where $\gamma = \pi U$) then $(Q,\pi) \overset{G\circ F}{\sim} D$

<u>Proof</u>. Let Π and Γ be the matrices induced by the vectors π and γ and the partitions F and G. Let $\hat{\Pi}$ be the matrix induced by π and the partition $G\circ F$. First of all, a direct calculation yields $\Gamma\Pi = \hat{\Pi}$. Also

$\underset{\sim}{U}_F\underset{\sim}{U}_G = \underset{\sim}{U}_{G\circ F}$ where $\underset{\sim}{U}_F$, $\underset{\sim}{U}_G$ and $\underset{\sim}{U}_{G\circ F}$ are the summing matrices associated with the partitions F, G and $G\circ F$. Since $(Q,\pi) \overset{F}{\sim} Y$, $\forall n,t_1,t_2,\cdots,t_n$, $\Pi Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U}_F = Y(t_1)Y(t_2)\cdots Y(t_n)$. Also, since $(Y,\gamma) \overset{G}{\sim} D$, $\Gamma Y(t_1)Y(t_2)\cdots Y(t_n)\underset{\sim}{U}_G = D(t_1)D(t_2)\cdots D(t_n)$. Thus

$\Gamma\Pi Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U}_F\underset{\sim}{U}_G = D(t_1)D(t_2)\cdots D(t_n)$. Thus

$\widehat{\Pi}Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U}_{G\circ F} = D(t_1)D(t_2)\cdots D(t_n)$, so $(Q,\pi) \overset{G\circ F}{\sim} D$. □

7. Equivalence Between two MRP's.

In section 6, equivalence between a MRP and a renewal process was generalized to collapsibility between two MRP's. The results in that section will prove to be useful in the next two chapters. In view of (2.6.1) and (2.6.2), though, it seems that the real meaning of $(Q,\gamma) \overset{F}{\sim} Y$ is cloudy. A more intuitively appealing generalization of equivalence between a MRP and a renewal process is given by

Definition 2.7.1. Let (Q,γ) and (Y,β) be two MRP's. We say (Q,γ) is equivalent to (Y,β) (written $(Q,\gamma) \sim (Y,\beta)$) if

$$\forall n,t_1,t_2,\cdots,t_n \quad \gamma Q(t_1)Q(t_2)\cdots Q(t_n)U = \beta Y(t_1)Y(t_2)\cdots Y(t_n)U.$$

Let (Q,γ) be $\{X_n, T_n\}$ and let (Y,β) be $\{Z_n, S_n\}$. If $(Q,\gamma)\sim(Y,\beta)$ then the distribution of $\{T_1,T_2,\cdots\}$ is the same as the distribution of $\{S_1,S_2,\cdots\}$. Strong lumpability, weak lumpability and collapsibility are not really equivalence relations between MRP's since the relation is always from the bigger one to the smaller one. Equivalence is an equivalence relation between MRP's. It is also the condition that must be satisfied by two random processes if one is to be substituted for the other in (say) a queueing system. Furthermore, it is a weaker condition than strong lumpability, weak lumpability and collapsibility.

Theorem 2.7.1. Let Π and $\underset{\sim}{U}$ be given by (1.2.1) and (1.2.2). If $(Q,\pi) \overset{F}{\sim} Y$ then $(Q,\pi) \sim (Y,\beta)$ where $\beta = \pi\underset{\sim}{U}$.

Proof. $(Q,\pi) \overset{F}{\sim} Y$ implies

$$\forall n, t_1, t_2, \cdots, t_n, \quad \Pi Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U} = Y(t_1)Y(t_2)\cdots Y(t_n).$$

If $\beta = \pi\underset{\sim}{U}$ then $\beta\Pi = \pi\underset{\sim}{U}\Pi = \pi$, so

$$\begin{aligned} \pi Q(t_1)Q(t_2)\cdots Q(t_n)U &= \beta\Pi Q(t_1)Q(t_2)\cdots Q(t_n)\underset{\sim}{U}U \\ &= \beta Y(t_1)Y(t_2)\cdots Y(t_n)U. \end{aligned}$$

Thus, $(Q,\pi) \sim (Y,\beta)$. $\square$

CHAPTER III

EQUIVALENCE BETWEEN CANONICAL MRP'S

1. The Core

Theorem 2.2.7 states that $(Q,\gamma) \sim r$ if and only if $\gamma \in \mathcal{K}_\gamma$ where $\mathcal{K}_\gamma$ is the largest set of probability vectors satisfying

$$\mathcal{K}_\gamma Q(t) \subset r(t)\,\mathcal{K}_\gamma, \quad \forall t \quad \text{and} \quad \beta Q(t)U = \gamma Q(t)U, \ \forall \beta \in \mathcal{K}_\gamma.$$

By Corollary 2.2.5.1, if Q is irreducible and $(Q,\gamma) \sim r$ then $(Q,\pi) \sim r$ where π is the steady state vector for Q. Thus, if $(Q,\gamma) \sim r$ and $(Q,\beta) \sim r$ then $\mathcal{K}_\gamma = \mathcal{K}_\pi$ and $\mathcal{K}_\beta = \mathcal{K}_\pi$ so $\mathcal{K}_\gamma = \mathcal{K}_\beta$. This leads to the following definition.

Definition 3.1.1. Let Q be an irreducible MRP with steady state vector π. The core of Q is the set $\mathcal{K}_\pi$. It will be denoted $\mathcal{K}$.

If $\mathcal{K} = \phi$ then Q is not equivalent to any renewal process. For any probability vector γ, either $\mathcal{K}_\gamma = \phi$ or $\mathcal{K}_\gamma = \mathcal{K}$.

Theorem 3.1.1. If Q is irreducible with steady state π and $Q \sim r$ then $r(t) = \pi Q(t)U$.

Proof. $Q \sim r$ implies that for some γ, $(Q,\gamma) \sim r$. Since Q is irreducible $\gamma \in \mathcal{K}$ so $\gamma Q(t)U = \pi Q(t)U$, $\forall t$. Thus $r(t) = \pi Q(t)U$.□

In general it is very difficult to determine what $\mathcal{K}$ is unless it can be shown to be empty or contain a single vector. Clearly, if $\mathcal{K}$ consists of a single vector, it must be the steady state vector. If $Q \sim r$ where Q is a finite state, irreducible MRP, one can find a set that must be inside the core.

<u>Theorem</u> 3.1.2. Let Q be a finite and irreducible and assume $Q \sim r$. Let $\mathscr{T} \subset [0,\infty]$ be the set $\{t\colon\ Q(t) \text{ is irreducible}\}$. For each $t \in \mathscr{T}$ let γ_t be the unique probability vector that satisfies $\gamma_t Q(t) = r(t)\gamma_t$ and let $\mathscr{W}$ be the linear subspace generated by $\{\gamma_t\}$, $t \in \mathscr{T}$. Then $\mathscr{W} \cap \mathscr{P} \subset \mathscr{K}$.

<u>Proof</u>. By Lemma 2.2.5, $\mathscr{K}$ is convex, compact and invariant under multiplication by $Q(t)/r(t)$. Thus there exists $\gamma_t \in \mathscr{K}$ that satisfies $\gamma_t Q(t)/r(t) = \gamma_t$ by the Brouwer Fixed Point Theorem. Thus, if $\beta \in \mathscr{W}$ then $\beta Q(t)U = (\beta U)r(t)$ since β is a linear combination of elements of $\mathscr{K}$, and $\beta Q(t)/r(t) \in \mathscr{W}$ since it too is a linear combination of elements of $\mathscr{K}$. Thus $(\mathscr{W} \cap \mathscr{P})Q(t)/r(t) \subset (\mathscr{W} \cap \mathscr{P})$ which implies that $\mathscr{W} \cap \mathscr{P} \subset \mathscr{K}$.

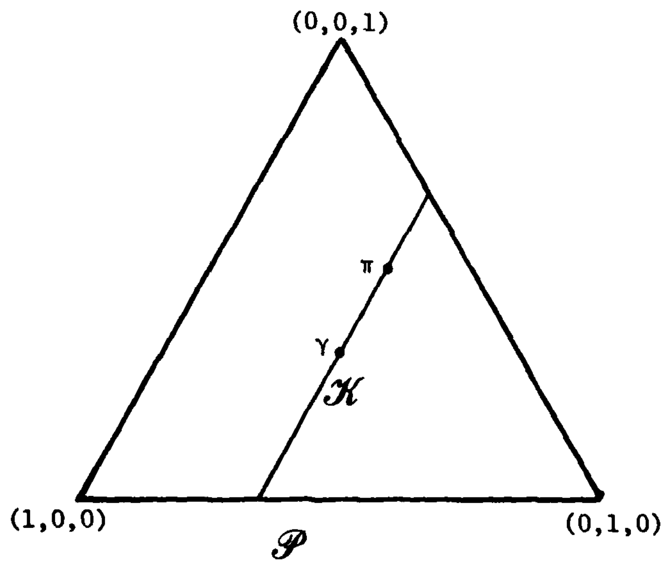
<u>Example</u> 3.1.1. We continue example 2.2.1. Recall

$$Q(t) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{if } t < t^* \\[2ex] \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \\[1ex] \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\[1ex] \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} & \text{if } t^* \le t < t^{**} \\[2ex] \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\[1ex] \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\[1ex] \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} & \text{if } t \ge t^{**}, \end{cases}$$

$$\gamma = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \quad \pi = (\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}), \quad r(t) = \begin{cases} 0 & \text{if } t < t^* \\ \tfrac{1}{2} & \text{if } t^* \le t < t^{**} \\ 1 & \text{if } t \ge t^{**}. \end{cases}$$

Note that $\gamma Q(t) = r(t)\gamma$ if $t^* \le t < t^{**}$ and π is the steady state vector. Since $Q \sim r$ we know that both γ and π must be in $\mathcal{K}$ (this was also proven in example 2.2.1).

Consider the vector $\beta = (1,0,0)$, which is not a linear combination of γ and π. Since $\beta Q(t)U = \frac{5}{12}$ when $t^* \le t < t^{**}$, $\beta \notin \mathcal{K}$. By theorem 3.1.2, the intersection of the line through γ and π and $\mathcal{P}$ is in $\mathcal{K}$. Since $\beta \notin \mathcal{K}$, $\mathcal{K}$ cannot be two dimensional. Thus $\mathcal{K}$ is the line through γ and π on $\mathcal{P}$.



If Q is not irreducible there is no unique steady state vector so one must be careful when talking about equivalence and about the core.

2. Reducible MRP's

The main problem with reducible MRP's is that they can be equivalent to different renewal processes depending on the initial

distribution. For example, say $(Q_1,\pi_1) \sim r_1$ and $(Q_2,\pi_2) \sim r_2$. If $\pi = (\pi_1,0)$, $\pi' = (0,\pi_2)$ and

$$Q(t) = \begin{array}{|c|c|}\hline Q_1(t) & \\ \hline & Q_2(t) \\ \hline\end{array}$$

then $(Q,\pi) \sim r_1$ and $(Q,\pi') \sim r_2$.

Another problem with reducible MRP's is that in general they contain some transient components. Recall the canonical form of a kernel (1.3.1). The states corresponding to the $A_i(t)$'s are irreducible sets of recurrent states and the states corresponding to the $B_i(t)$'s are irreducible sets of transient states. Let $\mathscr{T}$ be the set of states corresponding to the B_i's. Let $Q(t)$ have the form (1.3.1).

<u>Theorem</u> 3.2.1. If $(Q,\gamma) \sim r$ then there is a probability vector γ' such that $\gamma'_j = 0$, $\forall j \in \mathscr{T}$ and $(Q,\gamma') \sim r$.

<u>Proof</u>. Globally, $Q(\infty)$ has the form

$$\begin{array}{|c|c|}\hline A & \\ \hline C & B \\ \hline\end{array} \tag{3.2.1}$$

where A and B are square matrices, and $Q^n(\infty)$ has the form

$$\begin{array}{|c|c|}\hline A^n & \\ \hline C_n & B^n \\ \hline\end{array}\ .$$

Since B is a transient MRP, $B^n \to 0$ pointwise and $\lim C_n$ is well defined. Thus

$$\lim_{n\to\infty} (\gamma Q^n(\infty))_j = 0, \quad \forall j \in \mathscr{T}.$$

By theorem 2.2.7, $(Q,\gamma) \sim r$ implies $\gamma \in \mathscr{K}_\gamma$. Since $\mathscr{K}_\gamma$ is closed, convex and invariant under multiplication by $Q(\infty)$ we have

$$\gamma^* = \lim_{n\to\infty} \frac{1}{m} \sum_{k=n}^{n+m-1} \gamma Q^k(\infty) \in \mathscr{K}_\gamma$$

where m is the period of the states in A (i.e. the least common multiple of the periods of the A_i's). γ^* satisfies $\gamma_j^* = 0$, $\forall j \in \mathscr{T}$ and $(Q,\gamma^*) \sim r$. □

Corollary 3.2.1.1. Let Q have the form 1.3.1 where A is k x k. If $(Q,\gamma) \sim r$ then $(Q,\gamma') \sim r$ where γ' is made up of the first k entries of γ^*.

Proof. From the theorem, $(Q,\gamma) \sim r$ implies $(Q,\gamma^*) \sim r$ where $\gamma_j^* = 0$, $\forall j \in \mathscr{T}$. Thus,

$$\forall n, t_1,t_2,\cdots,t_n, \quad \gamma^* Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n).$$

But $\gamma^* Q(t_1)Q(t_2)\cdots Q(t_n)U = \gamma' A(t_1)A(t_2)\cdots A(t_n)U$, so $(Q,\gamma') \sim r$. □

In essence, we have shown that as far as equivalence to a renewal process is concerned, transient components are irrelevant. We will return to MRP's with transient components in a different context in section 5.

A general MRP without transient components has a kernel of the form

$$(3.2.2) \qquad Q(t) = \left[\begin{array}{c|c|c|c} Q_1(t) & & & \\ \hline & Q_2(t) & & \\ \hline & & \ddots & \\ \hline & & & Q_m(t) \end{array}\right]$$

where $Q_j(\infty)$ is an irreducible $n_j \times n_j$ stochastic matrix.

<u>Theorem</u> 3.2.2. Let $\gamma = (p_1\gamma_1, p_2\gamma_2, \cdots, p_m\gamma_m)$ where γ_j has n_j elements, $\gamma_j U = 1$, $\forall j$, and $\sum_{i=1}^{m} p_i = 1$. Let $p = (p_1, p_2, \cdots, p_m)$ and

$$Y(t) = \left[\begin{array}{c|c|c|c} r_1(t) & & & \\ \hline & r_2(t) & & \\ \hline & & \ddots & \\ \hline & & & r_m(t) \end{array}\right].$$

Then $(Q,\gamma) \sim Y$ if and only if $(Q_j,\gamma_j) \sim r_j$ for each j.

<u>Proof</u>. ($\Longleftarrow$) If $(Q_j,\gamma_j) \sim r_j$ then the core, $\mathscr{K}_j$ of Q_j contains γ_j. Let

$$\Gamma = \begin{bmatrix} \boxed{\gamma_1} & & & \\ & \boxed{\gamma_2} & & \\ & & \ddots & \\ & & & \boxed{\gamma_m} \end{bmatrix},$$

and let

$$\underset{\sim}{\mathscr{K}} = \begin{bmatrix} \mathscr{K}_1 & & & \\ & \mathscr{K}_2 & & \\ & & \ddots & \\ & & & \mathscr{K}_m \end{bmatrix} .$$

Since $\Gamma \in \underset{\sim}{\mathscr{K}}$, $\underset{\sim}{\mathscr{K}}Q(t) \subset Y(t)\underset{\sim}{\mathscr{K}}$, $\forall t$ and $Y(t) = \Gamma Q(t)\underset{\sim}{U}$, we have $(Q,\gamma) \sim Y$. Note that the starting vector for Y is $\gamma\underset{\sim}{U} = p$.

($\Longrightarrow$) First, $p_j > 0$, $\forall j$ since if not $r_j(t) = 0$, $\forall t$. Thus, since $(Q,\gamma) \sim Y$ there must be an invariant set of matrices $\underset{\sim}{\mathscr{K}}$ with $\Gamma \in \underset{\sim}{\mathscr{K}}$ such that $\underset{\sim}{\mathscr{K}}Q(t) \subset Y(t)\underset{\sim}{\mathscr{K}}$, $\forall t$. Let $\mathscr{K}_j$ be the set of vectors that correspond to the j^{th} row of the matrices in $\underset{\sim}{\mathscr{K}}$. Thus $\mathscr{K}_j Q_j(t) \subset r_j(t)\mathscr{K}_j$, $\forall t$. Since $\gamma_j \in \mathscr{K}_j$, this implies $(Q_j,\gamma_j) \sim r_j$. □

The main result of this section says that if a reducible MRP is equivalent to a renewal process, then every component (with positive initial probability) must be equivalent to that same renewal process.

<u>Theorem</u> 3.2.3. Let $Q(t)$ have the form (3.2.2) and let π_j be the unique vector satisfying $\pi_j Q_j(\infty) = \pi_j$, $\pi_j U = 1$. Let $\pi = (p_1\pi_1, p_2\pi_2, \cdots, p_m\pi_m)$ where $\sum_{j=1}^{m} p_j = 1$. Then $(Q,\pi) \sim r$ if and only if for each j either $(Q_j,\pi_j) \sim r$ or $p_j = 0$.

<u>Proof</u>. ($\Longleftarrow$) Let $j_1, j_2, \cdots, j_k$ be the set $\{j: p_j > 0\}$. Clearly, $(Q,\pi) \sim Y$ where

$$Y(t) = \begin{bmatrix} Q_{j_1}(t) & & & \\ & Q_{j_2}(t) & & \\ & & \ddots & \\ & & & Q_{j_k}(t) \end{bmatrix} .$$

By theorem 3.2.2, $(Y,\pi') \sim Z$ where $\pi' = (p_{j_1}\pi_{j_1}, p_{j_2}\pi_{j_2}, \cdots, p_{j_k}\pi_{j_k})$ and

$$Z(t) = \begin{array}{|c|c|c|c|} \hline r(t) & & & \\ \hline & r(t) & & \\ \hline & & \ddots & \\ \hline & & & r(t) \\ \hline \end{array} \;.$$

Since Z is strongly lumpable to r we have $(Z, p') \sim r$ where $p' = (p_{j_1}, p_{j_2}, \cdots, p_{j_k})$. Thus, $(Q,\pi) \sim Y$, $(Y,\pi') \sim Z$ and $(Z,p') \sim r$. By theorem 2.7.6 we have $(Q,\pi) \sim r$.

($\Longrightarrow$) If $(Q,\pi) \sim r$ then $\forall n$, $t_1, t_2, \cdots, t_n$, $\pi Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$. By the form of $Q(t)$,

$$\pi Q(t_1)Q(t_2)\cdots Q(t_n)U = \sum_{j=1}^{m} p_j \pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)U.$$

For any N,

$$\pi Q(t_1)Q(t_2)\cdots Q(t_n)Q^N(\infty)Q(t_1)Q(t_2)\cdots Q(t_n)U = (r(t_1)r(t_2)\cdots r(t_n))^2.$$

Thus, for any N,

$$\sum_{j=1}^{m} p_j \pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)Q_j^N(\infty)Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)U$$
$$= (r(t_1)r(t_2)\cdots r(t_n))^2.$$

Say $Q_j(\infty)$ is periodic with period k_j. We have for any N,

$$\sum_{j=1}^{m} p_j \pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)\left[\frac{1}{k_j}\sum_{i=N}^{N+k_j-1} Q_j^i(\infty)\right]Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)U$$
$$= (r(t_1)r(t_2)\cdots r(t_n))^2.$$

Thus,

$$\sum_{j=1}^{m} p_j \pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)\left[\lim_{n\to\infty} \frac{1}{k_j} \sum_{i=N}^{N+k_j-1} Q_j^i(\infty)\right] Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)U$$
$$= (r(t_1)r(t_2)\cdots r(t_n))^2.$$

Let $\pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n) = a_j\gamma_j$ where $\gamma_j U = 1$. Since

$\gamma_j\left[\lim_{N\to\infty} \frac{1}{k_j} \sum_{i=N}^{N+k_j-1} Q_j^i(\infty)\right] = \pi_j$ we have

$$\sum_{j=1}^{m} p_j \pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)\left[\lim_{N\to\infty} \frac{1}{k_j} \sum_{i=N}^{N+k_j-1} Q_j^i(\infty)\right] Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)U$$
$$= \sum_{j=1}^{m} p_j a_j^2.$$

Thus,

$$\sum_{j=1}^{m} p_j a_j^2 = (r(t_1)r(t_2)\cdots r(t_n))^2,$$

and

$$\sum_{j=1}^{m} p_j a_j = r(t_1)r(t_2)\cdots r(t_n),$$

so

$$\sum_{j=1}^{m} p_j a_j^2 - \left[\sum_{j=1}^{m} p_j a_j\right]^2 = 0.$$

This term says that the variance of some random variables that take values a_j with probability p_j is zero. The only way this can be is if there is some common value, a, such that for each j either $a_j = a$ or $p_j = 0$.

Thus $r(t_1)r(t_2)\cdots r(t_n) = \sum_{j=1}^{m} p_j a_j = a$. This says that if $p_j > 0$ then $\pi_j Q_j(t_1)Q_j(t_2)\cdots Q_j(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$. Since $n, t_1, t_2, \cdots, t_n$ were arbitrary, we have $(Q_j, \pi_j) \sim r$. □

Theorem 3.2.3 is not the strongest result imaginable on this topic. A stronger result would be

Conjecture 3.2.1. If Q has form (3.2.2) them $(Q,\gamma) \sim r$ (where $\gamma = (p_1\gamma_1, p_2\gamma_2,\cdots,p_n\gamma_n)$) if and only if for each j either $(Q_j,\gamma_j) \sim r$ or $p_j = 0$.

Although $(Q_j,\gamma_j) \sim r$ or $p_j = 0$ for all j clearly implies $(Q,\gamma) \sim r$, the idea behind the proof of theorem 3.2.3 cannot be used to prove the converse. The problem is that in general we cannot force the distribution on the state space back to γ once it leaves it. We can force it back to π by looking far enough into the future. We can say something slightly stronger than theorem 3.2.3 though.

Corollary 3.2.3.1. If $(Q,\gamma) \sim r$ then for each j either $(Q_j,\pi_j) \sim r$ or $p_j = 0$.

Proof. Since $(Q,\gamma) \sim r$, $(Q,\pi) \sim r$ where $\pi = \lim_{n\to\infty} \gamma \sum_{j=n}^{n+m} Q^j(\infty)$ and m is the period of Q. The result follows from the theorem. □

The theorems in this section can be reformulated in an enlightening way if we extend the notion of a core to reducible MRP's.

Definition 3.2.1. A set $\mathcal{K}$ will be called a core of a MRP, Q, if there is some renewal process r such that $\mathcal{K}Q(t) \subset r(t)\mathcal{K}$, $\forall t$, and $\mathcal{K}$ is the largest set with that property.

If Q is irreducible the core is unique. If Q is not irreducible there may be several cores. For example, say

$$Q(t) = \begin{array}{|c|c|}\hline Q_1(t) & \\ \hline & Q_2(t) \\ \hline\end{array}$$

where $Q_1(t)$ and $Q_2(t)$ are irreducible and there exist cores $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1 Q_1(t) \subset r_1(t)\,\mathcal{K}_1$ and $\mathcal{K}_2 Q_2(t) \subset r_2(t)\,\mathcal{K}_2$, $\forall t$ (i.e. $Q_1 \sim r_1$ and $Q_2 \sim r_2$). Then $(\mathcal{K}_1, 0)$ and $(0, \mathcal{K}_2)$ are both cores of Q yielding different renewal processes.

<u>Theorem</u> 3.2.4. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are different cores of Q then $\mathcal{K}_1 \perp \mathcal{K}_2$.

<u>Proof</u>. Say $\mathcal{K}_1 Q(t)U = r_1(t)$ and $\mathcal{K}_2 Q(t)U = r_2(t)$ where $r_1 \neq r_2$. Let $\gamma = (p_1\gamma_1, p_2\gamma_2, \cdots, p_n\gamma_n) \in \mathcal{K}_1$ and $\beta = (q_1\beta_1, q_2\beta_2, \cdots, q_n\beta_n) \in \mathcal{K}_2$. Say $(p_j q_j)(\beta_j \bullet \gamma_j) \neq 0$ for some j. By theorem 3.2.3 this means $(Q_j, \pi_j) \sim r_1$ and $(Q_j, \pi_j) \sim r_2$ which is impossible. Thus either $p_j\beta_j = 0$ or $q_j\gamma_j = 0$ (or both), so $\gamma \bullet \beta = 0$. □

Let

$$Q(t) = \begin{array}{|c|c|c|c|} \hline Q_1(t) & & & \\ \hline & Q_2(t) & & \\ \hline & & \ddots & \\ \hline & & & Q_n(t) \\ \hline \end{array}\ ,$$

and suppose $\mathcal{K}$ is a core of Q yielding the renewal process r. Let $J \subset \{1,2,\cdots,n\}$ be the set of indices such that there exists some $\gamma \in \mathcal{K}$ with $p_j\gamma_j \neq 0$ $(\gamma = (p_1\gamma_1, p_2\gamma_2, \cdots, p_n\gamma_n))$. By corollary 3.2.3.1, $Q_j \sim r$, $\forall j \in J$. Let $\hat{\mathcal{K}}_j = (0, 0, \cdots, 0, \mathcal{K}_j, 0, 0, \cdots, 0)$ where $\mathcal{K}_j$ (the core of Q_j) is in the j^{th} spot. Let $\hat{\mathcal{K}}$ be the convex hull of $\bigcup_{j \in J} \hat{\mathcal{K}}_j$. In other words $\hat{\mathcal{K}}$ is all vectors of the form

$(p_1\gamma_1, p_2\gamma_2, \cdots, p_n\gamma_n)$ where $\sum_{j=1}^{n} p_j = 1$, $p_j = 0$ if $j \notin J$, and $\gamma_j \in \mathcal{K}_j$ if $j \in J$. Clearly $\hat{\mathcal{K}} \subset \mathcal{K}$.

Conjecture 3.2.2. $\hat{\mathcal{K}} = \mathcal{K}$.

Theorem 3.2.5. Conjecture 3.2.2 is true if and only if conjecture 3.2.1 is true.

Proof. $(\Longrightarrow)$ Say $\hat{\mathcal{K}} = \mathcal{K}$ and $(Q,\gamma) \sim r$ where $\gamma = (p_1\gamma_1, p_2\gamma_2, \cdots, p_n\gamma_n)$. Then $\gamma = \sum_{j\in J} p_j\gamma_j$ where $\gamma_j \in \mathcal{K}_j$. Thus $(Q_j,\gamma_j) \sim r$ whenever $p_j \neq 0$.

$(\Longleftarrow)$ If $(Q,\gamma) \sim r$ implies $(Q_j,\gamma_j) \sim r$ whenever $p_j \neq 0$ then $\gamma_j \in \mathcal{K}_j$ whenever $p_j \neq 0$. Thus any $\gamma \in \mathcal{K}$ is of the form $\sum_{j\in J} p_j\gamma_j$ where $\gamma_j \in \mathcal{K}_j$. □

3. Periodic MRP's

In this section we consider MRP's whose states are arranged so that the kernel has the form

$$(3.3.1) \qquad Q(t) = \left[\begin{array}{c|c|c|c|c} & Q_1(t) & & & \\ \hline & & Q_2(t) & & \\ \hline & & & \ddots & \\ \hline & & & & Q_{m-1}(t) \\ \hline Q_m(t) & & & & \end{array}\right]$$

where $Q_j(t)$ is $n_j \times n_{j+1}$ (interpret $j + 1$ as 1 if $j = m$), $1 < m < \infty$, $1 \leq n_j \leq \infty$, $j = 1,2,\cdots,m$, and $Q_j(t)$ is irreducible.

Such a MRP is said to have period m.

Since a MRP of the form (3.3.1) is irreducible, by theorem 3.1.1 there is a unique core $\mathcal{K}$, so if $(Q,\gamma) \sim r$ then $r(t) = \pi Q(t)U$ where π is the steady state vector.

Let $Q(t)$ have the form (3.3.1) and let π be the steady state vector for Q. Let $\pi = \frac{1}{m}(\pi_1, \pi_2, \cdots, \pi_m)$ where π_j has n_j elements and $\pi_j U = 1$. It might seem possible for $Q \sim r$ even if $\pi_j Q_j(t)U$ depends on j. Since one cannot be sure which phase the process starts in, r(t) could be an average of the random processes arising from each phase. It turns out this cannot occur.

Theorem 3.3.1. If $Q \sim r$ then $\pi_j Q_j(t)U = r(t)$, $\forall j$.

Proof. If $Q \sim r$ then $\forall n$, $t_1, t_2, \cdots t_n$, $\pi Q(t_1)Q(t_2)\cdots Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$. In particular, $\pi Q^{m-1}(\infty)Q(t_1)Q^{m-1}(\infty)Q(t_2)\cdots Q^{m-1}(\infty)Q(t_n)U = r(t_1)r(t_2)\cdots r(t_n)$. Thus $(\hat{Q},\pi) \sim r$ where $\hat{Q}(t) = Q^{m-1}(\infty)Q(t)$. But

$$\hat{Q}(t) = \begin{bmatrix} \hat{Q}_1(t) & & & \\ & \hat{Q}_2(t) & & \\ & & \ddots & \\ & & & \hat{Q}_m(t) \end{bmatrix},$$

where $\hat{Q}_j(t) = Q_j(\infty)Q_{j+1}(\infty)\cdots Q_m(\infty)Q_1(\infty)\cdots Q_{j-2}(\infty)Q_{j-1}(t)$.

By theorem 3.2.3 we must have $(\hat{Q}_j, \pi_j) \sim r$, $\forall j$. Thus

$\pi_j \hat{Q}_j(t)U = r(t),\ \forall j$. Since $\pi_j Q_j(\infty) = \pi_{j+1}$ we have

$\pi_j \hat{Q}_j(t)U = \pi_{j-1}Q_{j-1}(t)U$, so $\pi_j Q_j(t)U = r(t),\ \ \forall j$. □

The following results will be useful in the next chapter.

<u>Theorem</u> 3.3.2. Let $Q(t)$ have the form (3.3.1) and $Y(t)$ have the form

$$Y(t) = \left[\begin{array}{c|c|c|c} & r_1(t) & & \\ \hline & & \ddots & \\ \hline & & & r_{m-1}(t) \\ \hline r_m(t) & & & \end{array}\right].$$

Let $\pi = \frac{1}{m}(\pi_1, \pi_2, \cdots \pi_m)$ be the steady state vector for Q. If $\pi_j Q_j(t) = r_j(t)\pi_{j+1},\ \forall j$ then $(Q,\pi) \sim Y$.

<u>Proof</u>. We have $\Pi Q(t) = Y(t)\Pi$ where

$$\Pi = \begin{bmatrix} \pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_m \end{bmatrix}$$

so the result follows from theorem 2.7.2. □

<u>Theorem</u> 3.3.3. $(Q,\gamma) \sim Y$ if and only if there are sets $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_m$ of probability vectors in $\mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \cdots, \mathbb{R}^{n_m}$ respectively such that $\mathcal{K}_j Q_j(t) \subset r_j(t)\mathcal{K}_{j+1},\ j = 1,2,\cdots,m,\ \ \forall t$, and $\gamma_j \in \mathcal{K}_j$.

Proof. ($\Longleftarrow$) Let

$$\underset{\sim}{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_1 & & & \\ & \mathcal{K}_2 & & \\ & & \ddots & \\ & & & \mathcal{K}_m \end{bmatrix} .$$

Thus,

$$\underset{\sim}{\mathcal{K}}Q(t) = \begin{bmatrix} & \mathcal{K}_1 Q_1(t) & & \\ & & \ddots & \\ & & & \mathcal{K}_{m-1}Q_{m-1}(t) \\ \mathcal{K}_m Q_m(t) & & & \end{bmatrix} ,$$

and

$$Y(t)\underset{\sim}{\mathcal{K}} = \begin{bmatrix} & r_1(t)\mathcal{K}_2 & & \\ & & \ddots & \\ & & & r_{m-1}(t)\mathcal{K}_m \\ r_m(t)\mathcal{K}_1 & & & \end{bmatrix} .$$

Since $\mathcal{K}_j Q_j(t) \subset r_j(t)\,\mathcal{K}_{j+1}$, $\forall j$ we have $\underset{\sim}{\mathcal{K}}Q(t) \subset Y(t)\underset{\sim}{\mathcal{K}}$ so $(Q,\gamma) \sim Y$.

($\Longrightarrow$) Since $(Q,\gamma) \sim Y$ there is a set of matrices $\underset{\sim}{\mathcal{K}}$ satisfying $\underset{\sim}{\mathcal{K}}Q(t) \subset Y(t)\underset{\sim}{\mathcal{K}}$, $\forall t$. Let $\mathcal{K}_j$ be the set of vectors corresponding to the j^{th} row of $\underset{\sim}{\mathcal{K}}$. Let

$$\Gamma = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_m \end{bmatrix} ,$$

which is in $\underset{\sim}{\mathcal{K}}$. Thus,

$$\Gamma Q(t) = \begin{bmatrix} & \gamma_1 Q_1(t) & & \\ & & \ddots & \\ & & & \gamma_{m-1}Q_{m-1}(t) \\ \gamma_m Q_m(t) & & & \end{bmatrix} .$$

Since $\underset{\sim}{\mathcal{K}}Q(t) \subset Y(t)\underset{\sim}{\mathcal{K}}$ there is a $\Gamma' \in \underset{\sim}{\mathcal{K}}$

$$\Gamma' = \begin{bmatrix} \gamma_1' & & & \\ & \gamma_2' & & \\ & & \ddots & \\ & & & \gamma_m' \end{bmatrix} ,$$

such that $\Gamma Q(t) = Y(t)\Gamma'$. But

$$Y(t)\Gamma' = \begin{bmatrix} & r_1(t)\gamma_2' & & \\ & & \ddots & \\ & & & r_{m-1}(t)\gamma_m' \\ r_m(t)\gamma_1' & & & \end{bmatrix} .$$

Thus for any $\gamma_j \in \mathcal{K}_j$ there is a $\gamma_{j+1}' \in \mathcal{K}_{j+1}$ such that

$\gamma_j Q_j(t) = r_j(t)\gamma_{j+1}'$. This implies $\mathcal{K}_j Q_j(t) \subset r_j(t)\,\mathcal{K}_{j+1}$, $\forall j$, $\forall t$.

4. MRP's with transient states.

Let

$$(3.4.1) \qquad Q(t) = \begin{bmatrix} A(t) & 0 \\ C(t) & B(t) \end{bmatrix} ,$$

where $A(t)$ and $B(t)$ are square and irreducible and $C(t) \neq 0$. In section 2 it was shown that $Q \sim r$ if and only if $A \sim r$ so in the context of that section, the transient components could be ignored. If we examine the situation a little more closely it becomes apparent that MRP's with transient components can exibit some interesting properties of their own.

Let (Q,γ) be a MRP where $Q(t)$ has the form (3.4.1). Let $\gamma = (\gamma_1,\gamma_2)$ where γ_1 corresponds to A (the recurrent components) and γ_2 corresponds to B (the transient components). Clearly (B,γ_2) is a transient MRP. When equivalence between a MRP and a renewal process was defined there was nothing preventing the MRP from being transient. Thus, $(B,\gamma_2) \sim r$ if

$$\forall n,t_1,t_2,\cdots,t_n, \quad \gamma_2 B(t_1)B(t_2)\cdots B(t_n)U = r(t_1)r(t_2)\cdots r(t_n).$$

Lemma 3.4.1. If (B,γ_2) is a transient MRP and $(B,\gamma_2) \sim r$ then

(1) r is a transient renewal process (i.e. $r(\infty) < 1$).

(2) The lifetime of (B,γ_2) has the same distribution as the lifetime of r.

Proof. (1) By Theorem 2.2.8, $r(\infty)$ is the largest eigenvalue of $B(\infty)$ which is strictly less than one since B is transient.

(2) The lifetime of a MRP is the random variable $T_1 + T_2 + \cdots + T_{n^*}$, where $n^*(\omega) = \inf_n \{T_{n+1}(\omega) = \infty\}$. For the renewal process r, the distribution of n^* is given by $P(n^* = k) = r(\infty)^k\,(1 - r(\infty))$. For the MRP (B,γ_2), the distribution of n^* is given by $P(n^* = k) = \gamma_2 B(\infty)^k (I - B(\infty))U$. Since $(B,\gamma_2) \sim r$, $\gamma_2 B(\infty)^k (I - B(\infty))U = r(\infty)^k (1 - r(\infty))$, so n^* has the same distribution for both processes. Furthermore, in both cases the distribution of the lifetime, $L(t)$, is given by the convolution of $r(t)$ with itself n^* times (i.e. $L(t) = (1 - r(\infty)) \sum_{k=1}^{\infty} r(\infty)^k r^{(k)}(t)$ where $r^{(k)}(t)$

is the k-fold convolution of $r(t)$ with itself). □

One of the interesting properties that MRP's with transient states can exibit is equivalence to a delayed renewal process.

<u>Definition</u> 3.4.1. Let $\{S_0, S_1, \cdots\}$ be a random process. If $\{S_1, S_2, \cdots\}$ is a renewal process and S_0 is independent of $\{S_1, S_2, \cdots\}$ then $\{S_0, S_1, \cdots\}$ is called a delayed renewal process. If $P(S_0 \leq t) = g(t)$ and $P(S_n \leq t) = r(t)$, $n = 1, 2, \cdots$, the delayed renewal process is denoted (g, r).

<u>Definition</u> 3.4.2. Let (Q, γ) be a MRP on $(\Omega, \mathscr{F}, P)$. If there is a stopping time n^* such that $\{T_{n^*+1}, T_{n^*+2}, \cdots\}$ is a renewal process with distribution $r(t)$ and $\{T_{n^*+1}, T_{n^*+2}, \cdots\}$ is independent of $\{T_1, T_2, \cdots, T_{n^*}\}$ then $(Q, \gamma) \sim (g, r)$, where $g(t) = P(T_1 + T_2 + \cdots + T_{n^*} \leq t)$.

<u>Theorem</u> 3.4.2. Let (Q, γ) be a MRP where $Q(t)$ has the form (3.4.1). Say $(A, \gamma_1) \sim r$ and A has core $\mathscr{K}$. If

$$\forall n, t_1, t_2, \cdots, t_n \quad \frac{\gamma_2 B(t_1) B(t_2) \cdots B(t_{n-1}) C(t_n)}{\gamma_2 B(t_1) B(t_2) \cdots B(t_{n-1}) C(t_n) U} \quad \varepsilon\, \mathscr{K}$$

then $(Q, \gamma) \sim (g, r)$ where $g(t)$ is the distribution of the first exit time from the set of transient states.

<u>Proof</u>. Let (Q, γ) be $\{X_n, T_n\}$ and let $n^* = \inf_n \{X_n \ \varepsilon\ A\}$. Clearly, n^* is a stopping time. Let γ_t' be the vector

$$\gamma_t'(i) = P(X_{n^*} = i \mid T_1 + T_2 + \cdots + T_{n^*} \leq t).$$

If $\gamma_t' \ \varepsilon\ \mathscr{K}$, $\forall t$ then $(T_1 + T_2 + \cdots + T_{n^*})$ is independent of $\{T_{n^*+1}, T_{n^*+2}, \cdots\}$ and $P(T_{n^*+1} \leq t_1, T_{n^*+2} \leq t_2, \cdots, T_{n^*+n} \leq t_n)$ $= r(t_1) r(t_2) \cdots r(t_n)$, so to prove that $(Q, \gamma) \sim (g, r)$ it suffices to show $\gamma_t' \ \varepsilon\ \mathscr{K}$, $\forall t$.

Let Γ_1 be the set of states in A, let Γ_2 be the set of states in B, and let $h_{n*}(t) = P(T_1 + T_2 + \cdots + T_{n*} \leq t)$. Thus

$$\gamma_t'(i) = P(X_{n*} = i,\ X_0 \in \Gamma_1 \mid T_1 + T_2 + \cdots + T_{n*} \leq t)$$

$$+ P(X_{n*} = i,\ X_0 \in \Gamma_2 \mid T_1 + T_2 + \cdots + T_{n*} \leq t)$$

$$= \frac{\gamma_1(i)}{h_{n*}(t)} + \frac{1}{h_{n*}(t)} [\gamma_2 \sum_{k=1}^{\infty} P(n^*=k) \int_0^{t-\Sigma t_j} \cdots \int_0^{t-t_1} \int_0^{t} B(dt_1)B(dt_2)$$

$$\cdots B(dt_{k-1}) C(t-\Sigma t_j)]_i$$

By hypothysis, this is

$$= \frac{\gamma_1(i)}{h_{n*}(t)} + \frac{1}{h_{n*}(t)} [\sum_{k=1}^{\infty} P(n^*=k) \int_0^{t-\Sigma t_j} \cdots \int_0^{t-t_1} \int_0^{t} \gamma_2 B(dt_1)B(dt_2)$$

$$\cdots B(dt_{k-1}) C(t-t_{k-1}) U \gamma_{t_1 t_2 \cdots t_{k-1}}(i)]$$

where $\gamma_{t_1 t_2 \cdots t_{k-1}} \in \mathcal{K}$. Since $\mathcal{K}$ is convex we have

$$\gamma_t'(i) = \frac{\gamma_1(i)}{h_{n*}(t)} + \frac{1}{h_{n*}(t)} \sum_{k=1}^{\infty} P(n^*=k) h_k(t) \gamma_k^*(i)$$

where $\gamma_k^* \in \mathcal{K}$. So

$$\gamma_t'(i) = \frac{\gamma_1(i) + [h_{n*}(t) - P(n^*=0)]\gamma^*(i)}{h_{n*}(t)}$$

where $\gamma^* \in \mathcal{K}$. Thus

$$\gamma_t' = \frac{\gamma_1 + [h_{n*}(t) - P(n^*=0)]\gamma^*}{h_{n*}(t)}$$

Since $\gamma_1 U = P(n^*=0)$ and $\gamma^* U = 1$ we get $\gamma_t' U = 1$. Thus $\gamma_t \in \mathcal{K}$. This proves that $(Q,\gamma) \sim (g,r)$ where $r(t) = \gamma_1 A(t) U$ and $g(t) = h_{n*}(t)$. □

<u>Example</u> 3.4.1. Say $(A,\gamma) \sim r$ and let

$$Q(t) = \left[\begin{array}{c|c} A(t) & 0 \\ \hline (1-p)f(t)\gamma & pf(t) \end{array}\right],$$

where $f(t) = 1-e^{-\lambda t}$. Let $\gamma^* = (q\gamma,\ (1-q))$, where $0 \le q \le 1$.

<u>Claim</u>. $(Q,\gamma^*) \sim (g,r)$ where $g(t) = q + (1-q)(1-e^{-(1-p)\lambda t})$.

<u>Proof</u>. Let T_0 be the time that the process first enters the set of recurrent states. Since $\forall n, t_1, t_2, \cdots, t_n$,

$$\frac{(1-q)p^{n-1}(1-p)f(t)^n\gamma}{(1-q)p^{n-1}(1-p)f(t)^n\gamma U} = \gamma \quad \varepsilon\, \mathscr{K},$$

the theorem shows that $(Q,\gamma^*) \sim (g,r)$ where $g(t) = P(T_0 \le t)$. But

$$P(T_0 \le t) = q + (1-q)(1-p)\sum_{j=1}^{\infty} p^j f^{(j)}(t) = q + (1-q)(1-e^{-(1-p)\lambda t}).$$

Thus, with probability q, (Q,γ^*) is a renewal process from time zero, and with probability 1-q there is an exponential delay (while the system is in the transient state) followed by a renewal process.

CHAPTER IV

CONVOLUTIONS OF MRP'S

1. Convolutions of MRP's

Let r_1 and r_2 be two renewal processes. The convolution of r_1 and r_2 is denoted $r_1 * r_2$ and is the renewal process with distribution

$$r_1 * r_2(t) = \int_0^t r_1(t - s) r_2(ds),$$

or equivalently

$$r_1 * r_2(t) = \int_0^t r_1(ds) r_2(t - s).$$

The interepoch times in $r_1 * r_2$ can be thought of as an r_1 epoch followed by an r_2 epoch. Clearly $r_1 * r_2 = r_2 * r_1$. Let $r_1, r_2, \cdots, r_n$ be renewal processes. The renewal process $r_1 * r_2 * \cdots * r_n$ is defined inductively by

$$r_1 * r_2 * \cdots * r_n(t) = \int_0^t r_1 * r_2 * \cdots * r_{n-1}(ds) r_n(t - s).$$

Definition 4.1.1. If r is a renewal process then $r^{(n)}$ is the n-fold convolution of r with inself defined inductively by

$$r^{(0)}(t) = 1$$

$$r^{(1)}(t) = r(t)$$

$$r^{(k+1)}(t) = \int_0^t r^{(k)}(ds) r(t - s).$$

Let Q_1 and Q_2 be m state MRP's $(1 < m < \infty)$. The convolution of Q_1 and Q_2, denoted $Q_1 * Q_2$ can be defined in an analogous manner.

$$Q_1 * Q_2(t) = \int_0^t Q_1(ds) Q_2(t - s).$$

Likewise, if $Q_1, Q_2, \cdots, Q_n$ are MRP with the same state space, $Q_1 * Q_2 * \cdots * Q_n$ is the MRP with kernel defined inductively

$$Q_1 * Q_2 * \cdots * Q_n(t) = \int_0^t Q_1 * Q_2 * \cdots * Q_{n-1}(ds) Q_n(t-s).$$

Definition 4.1.2. If Q is a MRP then $Q^{(n)}$ is the n-fold convolution of Q with itself defined inductively by

$$Q^{(0)}(t) = I$$

$$Q^{(1)}(t) = Q(t)$$

$$Q^{(n+1)}(t) = \int_0^t Q^{(n)}(ds) Q(t-s).$$

Again, the interepoch times in $Q_1 * Q_2$ can be thought of as a Q_1 epoch followed by a Q_2 epoch. In other words

$$[Q_1 * Q_2(t)]_{ij} = \sum_k \int_0^t [Q_1(ds)]_{ik} [Q_2(t-s)]_{kj}.$$

Note that the convolution only makes sense if the MRP's have the same state space, and that $Q_1 * Q_2$ and $Q_2 * Q_1$ are not necessarily the same MRP since matrix multiplication does not commute.

Notation. The n-fold convolution of Q with itself will be denoted $Q^{(n)}$, and the matrix $Q(t)$ raised to the n^{th} power will be denoted $Q^n(t)$.

To prove that the definition of convolution of MRP's makes sense we need the following lemma.

Lemma 4.1.1. Let $Q_1, Q_2, \cdots, Q_n$ be MRP's with identical state spaces. Every entry of $Q_1 * Q_2 * \cdots * Q_n(t)$ is a nonnegative, nondecreasing, right continuous function.

Proof. Since for each j, Q_j is a MRP, $Q_j(t)$ is nonnegative, nondecreasing and right continuous. Thus $Q_1 * Q_2 * \cdots * Q_n(t)$

$$= \int_0^t \int_0^{t-s_1} \cdots \int_0^{t-\sum_{k=1}^{n-1} s_k} Q_1(ds_1)Q_2(ds_2)\cdots Q_{n-1}(ds_{n-1})Q_n(t - \sum_{k=1}^{n-1} s_k) > 0$$

Assume $Q_1 * Q_2 * \cdots * Q_k(t)$ is nondecreasing for $k < n$. Then

$$Q_1 * Q_2 * \cdots * Q_n(t) = \int_0^t Q_1 * Q_2 * \cdots * Q_{n-1}(t-s)Q_n(ds)$$

$$= \int_0^{t-\Delta} Q_1 * Q_2 * \cdots * Q_{n-1}(t-s)Q_n(ds) + \int_{t-\Delta}^{t} Q_1 * Q_2 * \cdots * Q_{n-1}(t-s)Q_n(ds)$$

where $0 < \Delta < t$. By the induction hypothesis this is

$$\geq \int_0^{t-\Delta} Q_1 * Q_2 * \cdots * Q_{n-1}(t-\Delta-s)Q_n(ds) + \int_{t-\Delta}^{t} Q_1 * Q_2 * \cdots * Q_{n-1}(t-s)Q_n(ds)$$

$$\geq \int_0^{t-\Delta} Q_1 * Q_2 * \cdots * Q_{n-1}(t-\Delta-s)Q_n(ds) = Q_1 * Q_2 * \cdots * Q_n(t-\Delta).$$

Assume $Q_1 * Q_2 * \cdots * Q_k$ is right continuous for $k < n$. Let $\{t_j\}$ $j = 1,2,\cdots$ be a decreasing sequence with $\lim_{j\to\infty} t_j = t$. To show $Q_1 * Q_2 * \cdots * Q_n(t)$ is right continuous it is sufficient to show that

$$\lim_{j\to\infty} Q_1 * Q_2 * \cdots * Q_n(t_j) = Q_1 * Q_2 * \cdots * Q_n(t).$$

$$\lim_{j\to\infty} Q_1 * Q_2 * \cdots * Q_n(t_j) = \lim_{j\to\infty} \int_0^{t_j} Q_1 * Q_2 * \cdots * Q_{n-1}(t_j-s)Q_n(ds)$$

$$= \lim_{j\to\infty} \int_0^{t} Q_1 * Q_2 * \cdots * Q_{k-1}(t_j-s)Q_n(ds) + \lim_{j\to\infty} \int_t^{t_j} Q_1 * Q_2 * \cdots * Q_{n-1}(t_j-s)Q_n(ds).$$

By the Monotone Convergence Theorem, this is

$$= \int_0^{t} \lim_{j\to\infty} Q_1 * Q_2 * \cdots * Q_{n-1}(t_j-s)Q_n(ds) + \lim_{j\to\infty} \int_t^{t_j} Q_1 * Q_2 * \cdots * Q_{n-1}(t_j-s)Q_n(ds).$$

Since $Q_1 * Q_2 * \cdots * Q_{n-1}(t)$ is right continuous,

$$\int_0^t \lim_{j\to\infty} Q_1 * Q_2 * \cdots * Q_{n-1}(t_j - s) Q_n(ds) = \int_0^t Q_1 * Q_2 * \cdots * Q_{n-1}(t-s) Q_n(ds)$$

$$= Q_1 * Q_2 * \cdots * Q_n(s).$$

Since $Q_1 * Q_2 * \cdots * Q_{n-1}(t)$ is nonnegative nondecreasing and bounded,

$$0 \le \lim_{j\to\infty} \int_t^{t_j} Q_1 * Q_2 * \cdots * Q_{n-1}(t_j - s) Q_n(ds)$$

$$< \lim_{j\to\infty} \int_t^{t_j} Q_1 * Q_2 * \cdots * Q_{n-1}(\infty) Q_n(ds)$$

$$= Q_1 * Q_2 * \cdots * Q_{n-1}(\infty) \lim_{j\to\infty} (Q_n(t_j) - Q_n(t)) = 0$$

Thus $\lim_{j\to\infty} Q_1 * Q_2 * \cdots * Q_n(t_j) = Q_1 * Q_2 * \cdots * Q_n(t)$. □

Lemma 4.1.2. $Q_1 * Q_2 * \cdots * Q_n(\infty) = Q_1(\infty) Q_2(\infty) \cdots Q_n(\infty)$.

Proof. Assume $Q_1 * Q_2 * \cdots * Q_k(\infty) = Q_1(\infty) Q_2(\infty) \cdots Q_k(\infty)$ for $k < n$.

Since $Q_1 * Q_2 * \cdots * Q_n(t)$ is right continuous and bounded,

$$Q_1 * Q_2 * \cdots * Q_n(\infty) = \lim_{t\to\infty} \int_0^t Q_1 * Q_2 * \cdots * Q_{n-1}(t-s) Q_n(ds)$$

$$= \lim_{t\to\infty} \int_0^\infty Q_1 * Q_2 * \cdots * Q_{n-1}(t-s) Q_n(ds) - \lim_{t\to\infty} \int_t^\infty Q_1 * Q_2 * \cdots * Q_{n-1}(t-s) Q_n(ds)$$

By the Monotone Convergence Theorem and the induction hypothesis,

$$\lim_{t\to\infty} \int_0^\infty Q_1 * Q_2 * \cdots * Q_{n-1}(t-s) Q_n(ds) = \int_0^\infty \lim_{t\to\infty} Q_1 * Q_2 * \cdots * Q_{n-1}(t-s) Q_n(ds)$$

$$= \int_0^\infty Q_1 * Q_2 * \cdots * Q_{n-1}(\infty) Q_n(ds)$$

$$= Q_1(\infty) Q_2(\infty) \cdots Q_{n-1}(\infty) \int_0^\infty Q_n(ds)$$

$$= Q_1(\infty) Q_2(\infty) \cdots Q_n(\infty).$$

Since $Q_1{*}Q_2{*}\cdots{*}Q_{n-1}(t)$ is nondecreasing and bounded

$$0 \le \lim_{t\to\infty} \int_t^\infty Q_1{*}Q_2{*}\cdots{*}Q_{n-1}(t-s)Q_n(ds)$$

$$\le \lim_{t\to\infty} \int_t^\infty Q_1{*}Q_2{*}\cdots{*}Q_{n-1}(\infty)Q_n(ds)$$

$$= Q_1(\infty)Q_2(\infty)\cdots Q_{n-1}(\infty) \lim_{t\to\infty} (Q(\infty) - Q(t)) = 0.$$

Thus $Q_1{*}Q_2{*}\cdots{*}Q_n(\infty) = Q_1(\infty)Q_2(\infty)\cdots Q_n(\infty)$. □

Corollary 4.1.2.1. If $Q_1,Q_2,\cdots,Q_n$ are persistent then $Q_1{*}Q_2{*}\cdots{*}Q_n(\infty)U = U$.

Proof. Since $Q_j(\infty)U = U$ for each j,

$$Q_1{*}Q_2{*}\cdots{*}Q_n(\infty)U = Q_1(\infty)Q_2(\infty)\cdots Q_n(\infty)U = U. \ \square$$

Lemmas 4.1.1, 4.1.2 and corollary 4.1.2.1 show that $Q_1{*}Q_2{*}\cdots{*}Q_n(t)$ is the kernel of a MRP. We also have

Corollary 4.1.2.2. If Q has steady state vector π then $Q^{(n)}$ also has steady state vector π.

Proof. $\pi Q^{(n)}(\infty) = \pi Q^n(\infty) = \pi$. □

In general, the canonical form of $Q_1{*}Q_2$ will not be the same as Q_1 or Q_2. In fact knowing just the canonical form of Q_1 and Q_2 is not enough to determine the canonical form of $Q_1{*}Q_2$. For example, say

$$Q_1(t) = \begin{pmatrix} 0 & 0 & X \\ X & X & 0 \\ X & X & 0 \end{pmatrix} \quad Q_2(t) = \begin{pmatrix} 0 & X & X \\ 0 & X & X \\ X & 0 & 0 \end{pmatrix}$$

(where X is a nonzero function of t). Q_1 and Q_2 are both irreducible and aperiodic but $Q_1{*}Q_2$ is reducible.

$$Q_1 * Q_2(t) = \begin{pmatrix} X & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{pmatrix}$$

Lemma 4.1.3. If Q is an irreducible, aperiodic, non-null MRP then so is $Q^{(n)}$.

Proof. Since Q is irreducible and non-null the steady state vector π has all entries positive. Also, $\lim_{k\to\infty} Q^k(\infty) = U\pi$. Thus, $\forall \varepsilon > 0$ and $\forall i,j$ there is a $k_{ij}(\varepsilon)$ such that

$$\forall k > k_{ij}(\varepsilon),\ |Q_{ij}{}^k(\infty) - \pi_j| < \varepsilon.$$

Choose $\varepsilon < \pi_i$ and k' large enough so that $k'n > k_{ii}(\varepsilon)$. Thus,

$$Q_{ii}^{k'n}(\infty) > 0 \text{ and } Q_{ii}^{k'(n+1)}(\infty) > 0.$$

Thus, the greatest common divisor of $\{k\colon [Q^{(n)}(\infty)]_{ii}^k > 0\}$ is one. Since this can be done for every state, $Q^{(n)}$ is aperiodic.

Choose $\varepsilon < \pi_j$ and k' large enough so that $k'n > k_{ij}(\varepsilon)$. Thus $[Q^{(n)}(\infty)]_{ij}^{k'} > 0$, so it is possible to get from state i to state j in $Q^{(n)}$. Since this can be done for any i and j, $Q^{(n)}$ is irreducible. □

2. Convolutions and Equivalence.

Ideally we would like $Q_1 \sim r_1, Q_2 \sim r_2, \cdots, Q_n \sim r_n$ to imply that $Q_1 * Q_2 * \cdots * Q_n \sim r_1 * r_2 * \cdots * r_n$. Unfortunately, the situation is not quite that simple. Consider the following example.

<u>Example</u> 4.2.1. Let,

$$Q_1(t) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{if } t < 1, \\[2ex] \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \end{pmatrix} & \text{if } 1 \le t < 2, \\[2ex] \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} & \text{if } t > 2, \end{cases}$$

$$Q_2(t) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{if } t < 1, \\[2ex] \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} & \text{if } 1 \le t < 2, \\[2ex] \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} & \text{if } t > 2. \end{cases}$$

By theorem 2.2.2, $Q_1 \sim r_1$ where $r_1(t) = \begin{cases} 0 & \text{if } t < 1, \\ \frac{1}{3} & \text{if } 1 \le t < 2, \\ 1 & \text{if } t \ge 2, \end{cases}$

and by example 2.2.1, $Q_2 \sim r_2$ where $r_2(t) = \begin{cases} 0 & \text{if } t < 1, \\ \frac{1}{2} & \text{if } 1 \le t < 2, \\ 1 & \text{if } t \ge 2. \end{cases}$

Direct calculations yield $r_1 * r_2(3) = \frac{2}{3}$ and

$$Q_1 * Q_2(3) = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{5}{18} \\ \frac{1}{6} & \frac{1}{6} & \frac{5}{18} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{18} \end{pmatrix} .$$

It turns out that $\frac{2}{3}$ is not an eigenvalue of $Q_1 * Q_2(3)$ so by theorem 2.2.8, $Q_1 * Q_2 \nsim r_1 * r_2$.

<u>Theorem</u> 4.2.1. If $Q \sim r$ then $Q^{(n)} \sim r^{(n)}$.

<u>Proof</u>. Let $\mathscr{K}$ satisfy $\mathscr{K} Q(t) \subset r(t)\mathscr{K}$, $\forall t$. Say $\mathscr{K} Q^{(k)}(t) \subset r^{(k)}(t)\mathscr{K}$, $\forall t$, $\forall k < n$. Then

$$\begin{aligned} \mathscr{K} Q^{(n)}(t) &= \mathscr{K} \int_0^t Q^{(n-1)}(t-s)Q(ds) \\ &\subset \int_0^t r^{(n-1)}(t-s)r(ds)\mathscr{K} \\ &= r^{(n)}(t)\mathscr{K} . \end{aligned}$$

Thus $\mathscr{K} Q^{(n)}(t) \subset r^{(n)}(t)\mathscr{K}$, so $Q^{(n)} \sim r^{(n)}$. $\square$

<u>Corollary</u> 4.2.1.1. If $(Q,\gamma) \sim r$ then $(Q^{(n)},\gamma) \sim r^{(n)}$.

<u>Proof</u>. Say $(Q^{(k)},\gamma) \sim r^{(k)}$ for $k < n$, and $\gamma Q^{(k)}(t) \in r^{(k)}(t)\mathscr{K}$, $\forall t$. Then,

$$\gamma Q^{(n)}(t) = \gamma \int_0^t Q^{(n-1)}(t-s)Q(ds)$$

$$\epsilon \int_0^t r^{(n-1)}(t-s)r(ds)\mathscr{K}$$

$$= r^{(n)}(t)\mathscr{K} .$$

Thus $(Q^{(n)},\gamma) \sim r^{(n)}$ and $\gamma Q^{(n)}(t)\ \epsilon\ r^{(n)}(t)\mathscr{K}$. □

Corollary 4.2.1.2. Let Q_j, $j = 1,2,\cdots,n$ be MRP's and assume there is a set $\mathscr{K}$ that satisfies $\mathscr{K}Q_j(t) \subset r_j(t)\mathscr{K}$, $\forall t$, $\forall j$. Then

$$Q_1*Q_2*\cdots*Q_n \sim r_1*r_2*\cdots*r_n.$$

Proof. By hypothesis, $Q_1 \sim r_1$. Say $Q_1*Q_2*\cdots*Q_k \sim r_1*r_2*\cdots*r_k$ and $\mathscr{K}Q_1*Q_2*\cdots*Q_k(t) \subset r_1*r_2*\cdots*r_k(t)\mathscr{K}$, $\forall t$, $\forall k < n$. Then

$$\mathscr{K}Q_1*Q_2*\cdots*Q_n(t) = \mathscr{K}\int_0^t Q_1*Q_2*\cdots*Q_{n-1}(t-s)Q_n(ds)$$

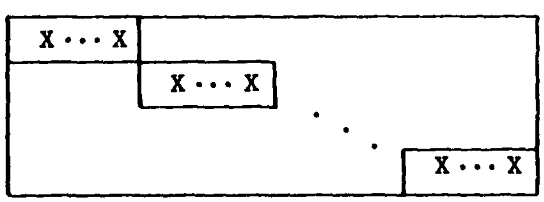
$$\subset \int_0^t r_1*r_2*\cdots*r_{n-1}(t-s)r_n(ds)\mathscr{K}$$

$$= r_1*r_2*\cdots*r_n(t)\mathscr{K} .$$

Thus, $Q_1*Q_2*\cdots*Q_n \sim r_1*r_2*\cdots*r_n$. □

Corollary 4.2.1.3. Let Q and Y be MRP's. If $Q \overset{F}{\sim} Y$ then $Q^{(n)} \overset{F}{\sim} Y^{(n)}$.

Proof. By theorem 2.6.4, $Q \overset{F}{\sim} Y$ implies there is a set of matrices, $\mathscr{M}$, of the form



(where each row is nonnegative and sums to one) so that $\mathcal{M}Q(t) \subset Y(t)\mathcal{M}$, $\forall t$. Say $Q^{(k)} \overset{F}{\sim} Y^{(k)}$ for $k < n$, and $\mathcal{M}Q^{(k)}(t) \subset Y^{(k)}(t)\mathcal{M}$, $\forall t$. Then

$$\begin{aligned}\mathcal{M}Q^{(n)}(t) &= \mathcal{M}\int_0^t Q^{(n-1)}(t-s)Q(ds) \\ &\subset \int_0^t Y^{(n)}(t-s)Y(ds)\mathcal{M} = Y^{(n)}(t)\mathcal{M},\end{aligned}$$

so $Q^{(n)} \overset{F}{\sim} Y^{(n)}$. □

The MRP $Q_1 * Q_2 * \cdots * Q_n$ can be analyzed in a different manner. Consider the MRP with kernel

$$Q(t) = \begin{array}{|c|c|c|c|c|}\hline & Q_1(t) & & & \\ \hline & & Q_2(t) & & \\ \hline & & & \ddots & \\ \hline & & & & Q_{n-1}(t) \\ \hline Q_n(t) & & & & \\ \hline\end{array}\;.$$

Let $\{A_1, A_2, \cdots, A_n\}$ be a partition of the states of Q where A_j is the collection of states corresponding to Q_j. Let f be the map that lumps the states of Q to $\{A_1, A_2, \cdots, A_n\}$ and let

$$Y(t) = \begin{array}{|c|c|c|c|c|}\hline & r_1(t) & & & \\ \hline & & r_2(t) & & \\ \hline & & & \ddots & \\ \hline & & & & r_{n-1}(t) \\ \hline r_n(t) & & & & \\ \hline\end{array} \; .$$

<u>Theorem</u> 4.2.2. If $Q \overset{F}{\sim} Y$ then $Q_1 * Q_2 * \cdots * Q_n \sim r_1 * r_2 * \cdots * r_n$.

<u>Proof</u>. From Corollary 4.2.1.3 we know that $Q^{(n)} \overset{F}{\sim} Y^{(n)}$. The form of Q and Y yields

$$Q^{(n)}(t) = \begin{array}{|c|c|c|c|}\hline Q_1 * Q_2 * \cdots * Q_n(t) & & & \\ \hline & Q_2 * Q_3 * \cdots * Q_1(t) & & \\ \hline & & \ddots & \\ \hline & & & Q_n * Q_1 * \cdots * Q_{n-1}(t) \\ \hline\end{array} \; ,$$

and

$$Y^{(n)}(t) = \begin{array}{|c|c|c|c|}\hline r_1 * r_2 * \cdots * r_n(t) & & & \\ \hline & r_2 * r_3 * \cdots * r_1(t) & & \\ \hline & & \ddots & \\ \hline & & & r_n * r_1 * \cdots * r_{n-1}(t) \\ \hline\end{array} \; .$$

The theorem follows by observing the top left components of $Q^{(n)}$ and $Y^{(n)}$

with theorem 3.2.2 in mind. □

Corollary 4.2.2.1. If $Q \overset{F}{\sim} Y$ then $\forall j$,

$Q_j * Q_{j+1} * \cdots * Q_{j-1} \sim r_1 * r_2 * \cdots * r_n$.

Proof. Since $r_j * r_{j+1} * \cdots * r_{j-1}(t) = r_1 * r_2 * \cdots * r_n(t)$, the remaining components of $Q^{(n)}$ and $Y^{(n)}$ give the result. □

Clearly, any sufficient condition for $Q \overset{F}{\sim} Y$ is sufficient for $Q_1 * Q_2 * \cdots * Q_n \sim r_1 * r_2 * \cdots * r_n$.

Theorem 4.2.3. If there exists $\gamma_1, \gamma_2, \cdots, \gamma_n$ such that $\forall j$, $\gamma_j Q_j(t) = r_j(t)\gamma_{j+1}$, $\forall t$, then $Q_1 * Q_2 * \cdots * Q_n \sim r_1 * r_2 * \cdots * r_n$,

Proof. Let

$$\Gamma = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \end{bmatrix} .$$

By hypothesis,

$$\Gamma Q(t) = \begin{bmatrix} & r_1(t)\gamma_2 & & \\ & & r_2(t)\gamma_3 & \\ & & & \ddots \\ & & & & r_{n-1}(t)\gamma_n \\ r_n(t)\gamma_{n-1} & & & & \end{bmatrix} = Y(t)\Gamma,$$

so $Q \overset{F}{\sim} Y$ by theorem 2.7.2. □

3. Markov-Renewal Functions and Markov-Renewal Equations.

Definition 4.3.1. Say Q is a MRP. Let $R_Q(t) = \sum_{n=0}^{\infty} Q^{(n)}(t)$. Similarly, if r is a renewal process then $R_r(t) = \sum_{n=0}^{\infty} r^{(n)}(t)$. $R_Q(t)$ is called the Markov-renewal function associated with Q and $R_r(t)$ is called the renewal function associated with r.

In this section we will only deal with irreducible, persistant MRP's. It can be shown [3] that $R_Q(t)$ is well defined and finite. Let (Q,γ) be defined on $(\Omega, \mathscr{F}, P)$, and let

$$N_{(Q,\gamma)}(t,\omega) = \sup \{n: \sum_{j=1}^{n} T_j(\omega) \leq t\}$$

$$N_{Q,\gamma}(t) = E(N_{(Q,\gamma)}(t,\cdot)).$$

Similarly, if r is defined on $(\Omega, \mathscr{F}, P)$ let $N_r(t)$ be defined analogously. For a MRP (or a renewal process) let us call the set of points $\{\sum_{j=0}^{n} T_j(\omega): n = 1,2,\cdots\}$, events. Thus $N_{(Q,\gamma)}(t)$ and $N_r(t)$ are the expected number of events in the interval $[0,t]$ for the corresponding random process (we assume there is an event at time zero). It is well known [3] that $N_r(t) = R_r(t)$ and $N_{(Q,\gamma)}(t) = \gamma R_Q(t)U$. Thus, theorem 4.2.1 has an important corollary.

Theorem 4.3.1. If $(Q,\gamma) \sim r$ then $N_{(Q,\gamma)}(t) = N_r(t)$.

Proof. Corollary 4.2.1.1 states that $(Q^{(n)},\gamma) \sim r$ for every n, so

$$\gamma Q^{(n)}(t)U = r^{(n)}(t).$$

Thus,

$$N_{(Q,\gamma)}(t) = \gamma R_Q(t)U = \gamma(\sum_{n=0}^{\infty} Q^{(n)}(t))U = \sum_{n=0}^{\infty} r^{(n)}(t) = R_r(t). \square$$

Actually, theorem 4.3.1 can be strengthened slightly. If Q has core $\mathscr{K}$ then $N_{(Q,\beta)}(t) = N_r(t)$ for every $\beta \in \mathscr{K}$.

Definition 4.3.2. Let Q be an n state MRP $(1 < n \leq \infty)$, and let $\hat{g}(t)$ be a column vector of n nonnegative functions bounded on finite intervals. An equation of the form

$$\hat{f}(t) = \hat{g}(t) + \int_0^t Q(ds)\hat{f}(t-s) \tag{4.3.1}$$

(where $\hat{f}(t)$ is an unknown column vector of functions) is called a Markov renewal equation.

If r is a renewal process and g(t) is a scalar function bounded on finite intervals then

$$f(t) = g(t) + \int_0^t r(ds)f(t-s) \tag{4.3.2}$$

(where f(t) is an unknown scalar function) is called a renewal equation.

The solution of (4.3.2) is unique and is given by

$$f(t) = \int_0^t R_r(ds)g(t-s). \tag{4.3.3}$$

Lemma 4.3.2. The solution of (4.3.1) is unique and is given by

$$\hat{f}(t) = \int_0^t R_Q(ds)\hat{g}(t-s). \tag{4.3.4}$$

Proof. Since Q is irreducible and persistent the lemma follows from [3, Chap. 10]. □

The j^{th} element of $\hat{f}(t)$ can be thought of as the value of something at time t given that Q is in state j at time zero. Thus, if we start with initial distribution γ, the expected value is

$$\gamma\hat{f}(t) = \int_0^t \gamma R_Q(ds)\hat{g}(t-s). \tag{4.3.5}$$

Let π be the steady state vector for Q and assume $Q \sim r$. It would be nice if it were always true that

$$\pi f(t) = \int_0^t R_r(ds)(\pi g(t-s))$$

since this involves only scalar integration. Unfortunately the situation is not quite that simple.

Theorem 4.3.3. $\pi\hat{f}(t) = \int_0^t R_r(ds)(\pi\hat{g}(t-s))$ for all bounded nonnegative g if and only if $\pi R_Q(t) = R_r(t)\pi$, $\forall t$.

Proof. ($\Longleftarrow$). $\pi\hat{f}(t) = \int_0^t \pi R_Q(ds)\hat{g}(t-s) = \int_0^t R_r(ds)(\pi\hat{g}(t-s))$.

($\Longrightarrow$) Say there exists a t such that $\pi R_Q(t) \neq R_r(t)\pi$. Then for some j, $[\pi R_Q(t)]_j \neq R_r(t)\pi_j$. Since $R_Q(t)$ and $R_r(t)$ are right continuous, there exists $\varepsilon > 0$ such that

$$\int_t^{t+\varepsilon} [\pi R_Q(ds)]_j \neq \int_t^{t+\varepsilon} R_r(ds)\pi_j.$$

Define $\hat{g}'$ as follows:

$$\hat{g}_i'(t) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \quad 0 \leq t < \varepsilon, \\ 0 & \text{if } i = j, \quad t \geq \varepsilon. \end{cases}$$

Thus,

$$\pi\hat{f}(t+\varepsilon) = \int_0^{t+\varepsilon} R_r(ds)(\pi\hat{g}(t+\varepsilon-s)) = \int_t^{t+\varepsilon} R_r(ds)\pi_j$$

$$\neq \int_t^{t+\varepsilon} [\pi R_Q(ds)]_j = \int_0^{t+\varepsilon} \pi R_Q(ds)\hat{g}'(t+\varepsilon-s)$$

$$= \pi\hat{f}(t+\varepsilon),$$

which is a contradiction. □

Corollary 4.3.3.1. If $\pi Q(t) = r(t)\pi$, $\forall t$, then

$$\pi\hat{f}(t) = \int_0^t R_r(ds)(\pi\hat{g}(t-s)).$$

Proof. First we show that $\pi Q^{(n)}(t) = r^{(n)}(t)\pi$, $\forall t$, $\forall n$. Assume it is true for $k < n$. Then

$$\pi Q^{(n)}(t) = \pi\int_0^t Q^{(n-1)}(t-s)Q(ds) = \int_0^t r^{(n-1)}(t-s)r(ds)\pi = r^{(n)}(t)\pi.$$

Thus, $\pi R_Q(t) = \pi\sum_{n=0}^{\infty} Q^{(n)}(t) = \sum_{n=0}^{\infty} r^{(n)}(t)\pi = R_r(t)\pi$, so the result holds.□

Thus, if $\pi Q(t) = r(t)\pi$, $\forall t$, the solution of any Markov-renewal equation in steady state is the same as the solution of the corresponding renewal equation. Unfortunately, $\pi Q(t) = r(t)\pi$, $\forall t$, is not a necessary condition for $Q \sim r$.

Let $Z_x^{(n)}(t) = Q^{(n)}(x+t)$, and let $\mathcal{Z}$ be the ring of matrix valued functions generated by $\{Z_x^{(n)}(\cdot) : x \in \mathbb{R},\ n = 0,1,2,\cdots\}$ under the usual operations of matrix multiplication and addition. If Q has core $\mathcal{K}$, then for any $\gamma \in \mathcal{K}$ and $Z(\cdot) \in \overline{\mathcal{Z}}$ (closure of $\mathcal{Z}$ in the usual sense) we have $\gamma Z(t)U = \pi Z(t)U$, where π is the steady state vector for Q.

<u>Theorem</u> 4.3.4. If $Q \sim r$ and $\widehat{g}(t) = Z(t)U$ where $Z(\cdot) \in \overline{\mathcal{Z}}$ then $\forall \gamma \in \mathcal{K}$, $\gamma\widehat{f}(t) = \int_0^t R_r(ds)(\gamma\widehat{g}(t-s))$.

<u>Proof</u>. Since $\widehat{g}(t) = Z(t)U$ we have $\gamma\widehat{g}(t) = \gamma'\widehat{g}(t)$, $\forall \gamma, \gamma' \in \mathcal{K}$. Thus,

$$\gamma\widehat{f}(t) = \int_0^t \gamma R_Q(ds)\widehat{g}(t-s)$$

$$= \int_0^t \sum_{n=0}^{\infty} \gamma Q^{(n)}(ds)\widehat{g}(t-s).$$

Since $\gamma Q^{(n)}(ds) = r^{(n)}(ds)\gamma_n$, where $\gamma_n \in \mathcal{K}$ we have

$$\gamma\widehat{f}(t) = \int_0^t \sum_{n=0}^{\infty} r^{(n)}(ds)(\gamma_n\widehat{g}(t-s)) = \int_0^t R_r(ds)(\gamma\widehat{g}(t-s)).\ \square$$

Although it appears that theorem 4.3.4 imposes a large restriction on the number of Markov-renewal equations we can consider, it should become clear from the following examples that in many problems of interest, $\hat{g}(t)$ will have that form. In the following examples, $Q \sim r$ and Q has steady state vector π.

Example 4.3.1. Let

$\hat{P}_x(t)_i$ = Pr [no events in the interval $(t, t+x]$ | Q starts in state i].

Then

$$\hat{P}_x(t) = (I - Q(t+x))U + \int_0^t Q(ds)\hat{P}_x(t-s).$$

Since $(I - Q(t+x)) \in \mathcal{Z}$, we know that

$$\pi\hat{P}_x(t) = \int_0^t R_r(ds)\pi(1 - Q(t+x-s))U$$

$$= \int_0^t R_r(ds)(1 - r(t+x-s)).$$

Example 4.3.2. Let

$\hat{P}'_x(t)_i$ = Pr [exactly one event in $(t, t+x]$ | Q starts in state i].

Then,

$$\hat{P}'_x(t) = \int_t^{t+x} Q(ds)(I - Q(t+x-s))U + \int_0^t Q(ds)\hat{P}'_x(t-s).$$

Since $\int_t^{t+x} Q(ds)(I - Q(t-x-s)) \in \bar{\mathcal{Z}}$, we have

$$\pi\widehat{P}'_x(t) = \int_0^t R_{\underline{\pi}}(ds)\pi \int_{t-s}^{t+x-s} Q(du)(I - Q(t+x-u))$$

$$= \int_0^t R_r(ds) \int_{t-s}^{t+x-s} r(du)(1 - r(t+x-u)).$$

Example 4.3.3. Let $N(t,\omega)$ be the number of events up to time t (including the events at 0), and let $\{T_1(\omega), T_2(\omega), \cdots\}$ be the interepoch times. The forward recurrence time at t, $F_t(\omega)$ is defined to be the time until the next event starting at t (i.e. $F_t(\omega) = (\sum_{j=1}^{N(t,\omega)} T_j(\omega)) - t$). Let

$$\widehat{f}_x(t)_i = \Pr(F_t \leq x \mid Q \text{ starts in state } i).$$

Then

$$\widehat{f}_x(t) = (Q(t+x) - Q(t))U + \int_0^t Q(ds)\widehat{f}_x(t-s),$$

so

$$\pi\widehat{f}_x(t) = \int_0^t R_r(ds)(r(t+x-s) - r(t-s)).$$

Example 4.3.4. Let $B_t(\omega) = t - \sum_{j=1}^{N(t,\omega)-1} T_j(\omega)$, be the backwards recurrence time. Let

$$\widehat{f}'_x(t) = \Pr(B_t > x \mid Q \text{ starts in state } i).$$

Then

$$\widehat{f}'_x(t) = (I - Q(t))1_{\{t > x\}} + \int_0^t Q(ds)\widehat{f}'_x(t-s).$$

Since $(I - Q(t))1_{\{t>x\}} = (I - Q(t))Q^{(0)}(t-x) \in \mathcal{Z}$,

we have

$$\widehat{f'_x}(t) = \int_0^t R_r(ds)(1 - r(t-s))1_{\{t-s>x\}}$$

$$= \int_0^{t-x} R_r(ds)(1 - r(t-s)).$$

CHAPTER V

DISCUSSION

1. Summary

The study of equivalent MRP's has its roots in the study of functions of Markov chains. Burke and Rosenblatt [1] gave conditions for functions of a Markov chain to be Markov. This came to be know as strong and weak lumpability. Kemeny and Snell [8] gives the best discussion of this topic. It can also be found in Rosenblatt [14]. Serfozo [15], [16] showed that the concept of lumpability in Markov chains extends easily to MRP's. These concepts are apparently considered unimportant by the masses since there has been very little reference to them in the literature since 1972.

The reason for the lack of interest is probably that strong lumpability is too strong a condition to be useful, and nobody has ever considered in detail the idea of weak lumpability to a renewal process. What has been shown here is that these concepts, in an appropriate modified form, are important both in applications and in the foundational study of MRP's. It would seem that any thorough discussion of MRP's would contain a section discussing the idea of the "time" processes of two MRP's having the same distribution. This has not been the case. The first discussion of this concept is found here in Chapter II and III.

There is a better reason for studying equivalence than the intuitive feeling that it should be done. Equivalence is exactly the condition for being able to substitute one MRP for another in (say) a

queueing network without changing the character of the system.

Examples of the use of this sort of tactic for simplifying the analysis of queueing systems abound, although until now there has been no theory that encompasses all of them. The best known example is that the output from a steady state M/M/1 queue is a Poisson process (as opposed to the infinite state MRP by which it is most naturally characterized). More results of this form can be found in Disney et al. [4] and Laslett [9]. There are also results that show certain flow processes in Jackson networks are Poisson [12]. The value of these kinds of results should be clear. They make it possible to analyze systems that would otherwise be intractable. As a simple example, consider two queues in tandem where the first queue is M/D/1/1 and the second is $\cdot/M/1/\infty$. Since the output from the M/D/1/1 queue is a renewal process in steady state, the second queue is a $G/M/1/\infty$ queue, which is well understood. We can easily write down the steady state queue length distributions, waiting times, and other quantities of interest. This would be impossible if it were not known that the output from the M/D/1/1 queue is renewal.

From the results of Chapter II, one can quickly verify all of these known results. One can also settle some previously unsolved problems (example 2.4.4) and come up with curious and unexpected results such as in examples 2.4.5 and 2.4.6.

Section 2.4 has shown that the tools developed in Chapter II are useful for solving important problems, especially problems dealing with flows in queueing networks. The emphasis in this paper, though, was on the development of the theory of equivalent MRP's. More specifically, the emphasis was on equivalence between a finite or count-

able state MRP and a renewal process.

In Chapter II, necessary, sufficient and necessary and sufficient conditions were given for $(Q,\gamma) \sim r$. The ideas of collapsibility and equivalence between two finite or countable state MRP's were discussed and conditions for them were given.

In Chapter III the various canonical forms of MRP's were examined in detail. Reducible and periodic MRP's were analyzed in terms of their components. One interesting observation was that a weighted average of steady state MRP's can only be renewal if each MRP is equivalent to the same renewal process (Theorem 3.2.3). It is likely (though unproven) that the same conclusion holds even if the MRP's are not in steady state (conjectures 3.2.1, 3.2.2). It would also be very interesting if these conjectures turned out to be false.

For the general finite or countable state MRP, answers were given to the questions

(1) For a given Q, which r can satisfy $Q \sim r$?

(2) For a given Q and r which γ can satisfy $(Q,\gamma) \sim r$?

Also, equivalence between a MRP and a delayed renewal process was defined and conditions were established for it.

Chapter IV was a look at the idea of convolutions of MRP's. The goal was to determine when equivalence is preserved under convolution. It became obvious that the answer is not simple. It is easy to produce an example of $Q_1 \sim r_1$, $Q_2 \sim r_2$, $Q_1 * Q_2 \nsim r_1 * r_2$, although conditions can be given to assure $Q_1 * Q_2 \sim r_1 * r_2$. One computational trick is to realize that $Q_1 * Q_2 * \cdots * Q_n$ can be thought of as a periodic MRP where the Q_j's are stages. The solution of Markov renewal equations involves the convolution of a MRP with itself. One would hope that if $Q \sim r$, the

solution of a Markov renewal equation involving Q could be simplified. This idea is made precise, and although not all Markov renewal equations can be simplified, it is shown that a large and important class can be.

2. Extensions.

Consider a renewal stream of customers entering a queueing system of n exponential servers in tandem (each with infinite waiting room). There is no problem analyzing the first queue. It is a G/M/1 queue, and its properties are well known. Unfortunately, the behavior at the second queue is almost impossible to analyze since the output from a G/M/1 queue is not a renewal process. In fact, according to the definitions in this paper, it is not even a MRP. In order to get the distribution of the time between one departure and the next, one must know whether the queue is empty or not. If it is empty, one still needs to know the distribution of the time until the next arrival in order to get the distribution of the time until the next departure. The foward recurrence time, except for the Poisson process, is a nontrivial function of a continuous parameter, $t \in (0,\infty)$. Thus, the only Markov chain one can find embedded at departure points must keep track of the queue length and the time since the last arrival (i.e. the state space is $[0,\infty] \times \{0,1,2,\cdots\}$). The output from the second queue is even worse. Again, the state space must include the queue length and the time since the last arrival. Unfortunately, the foward recurrence time for the input to the second queue is the foward recurrence time of a MRP with state space $[0,\infty] \times \{0,1,2,\cdots\}$. Thus, the departure process from the second queue is described by a MRP with state space $[0,\infty] \times \{0,1,2,\cdots\} \times \{0,1,2,\cdots\}$. (i.e. must keep track of the queue length at the first and second queue, and how long it has been since the last arrival to the first queue). The state space for the

departure process from the n^{th} queue is $[0,\infty] \times \{0,1,2,\cdots\}^n$.

Although finding the steady state queue length distribution (or even approximating it) in this very simple queueing network is a very difficult problem, the fact is that if any method is developed to deal with this problem the same method could probably be used to solve for queue length distributions in an arbitrary network with independent renewal arrivals and servers. This is because in a network of that form the state space needed to describe the flow of traffic on any arc in the network can never be worse than $[0,\infty]^{n_1} \times \{0,1,2,\cdots\}^{n_2}$.

The concept of equivalence between a MRP and a renewal process can easily be extended to MRP's with general state space, and state spaces like $[0,\infty]^{n_1} \times \{0,1,2,\cdots\}^{n_2}$ pose no problem at all. Let $(S,\mathscr{S})$ be a measure space and let $(\Omega,\mathscr{F},P)$ be a probability space.

Let $X_n\colon \Omega \to S$ be $\mathscr{S}$-measurable and let $T_n\colon \Omega \to [0,\infty]$ be $\mathscr{B}$-measurable. Then $\{X_n, T_n\}$ is a MRP with state space $(S,\mathscr{S})$ if for each t there is a $Q_{sA}(t)\colon S \times \mathscr{S} \to [0,1]$ such that

$$P(X_n \in A, T_n \le t \mid X_{n-1}, X_{n-2}, \cdots, X_0, T_{n-1}, T_{n-2}, \cdots, T_1) = Q_{X_{n-1}A}(t) \quad \text{a.s.}$$

where $Q_{\cdot A}(t)$ is measurable for fixed A, and $Q_{s\cdot}(t)$ is a measure on $\mathscr{S}$ for fixed s. In queueing networks $S = [0,\infty]^{n_1} \times \{0,1,2,\cdots\}^{n_2}$ and $\mathscr{S} = \sigma\{\mathscr{B}^{n_1} \times \mathscr{N}^{n_2}\}$ where $\mathscr{N} = 2^{\{0,1,2,\cdots\}}$. The initial distribution of the MRP is a measure μ on $(S,\mathscr{S})$ where $\mu(A) = P(X_0 \in A)$. We can denote a MRP on $(S,\mathscr{S})$ with kernel $Q(t)$ and initial distribution μ by (Q,μ). Let r be a renewal process. We say $(Q,\mu) \sim r$ if

$$\forall n, t_1, t_2, \cdots, t_n, \quad P(T_1 \le t_1, T_2 \le t_2, \cdots, T_n \le t_n) = r(t_1)r(t_2)\cdots r(t_n)$$

which implies

$$\forall n, t_1, t_2, \cdots, t_n, \qquad \iint \cdots \int \mu(ds_0) Q_{s_0 ds_1}(t_1) Q_{s_1 ds_2}(t_2) \cdots Q_{s_{n-1} ds_n}(t_n)$$
$$= r(t_1) r(t_2) \cdots r(t_n).$$

It seems clear that the conditions and properties for equivalence in this more general context should be analogous to those given in this paper.

3. Generalizations.

There is at least one direction that the concept of equivalence between MRP's can be generalized that could prove to be invaluable in the applied study of random processes. Equivalence is the condition that enables one to substitute one MRP for another without changing the character of the system at all. If there were some metric on the space of MRP's, one would expect equivalent MRP's to be zero distance apart (i.e. the same point) in that space. The general notion referred to above is this metric on equivalence classes of MRP's. Ideally, this metric would be easy to compute and a large class of queues (thought of as operators on the space of MRP's) would be continuous operators.

First of all, it is possible to define a metric on the space of MRP's; for instance the Levy metric [10]

$$d((Q_1,\gamma_1),(Q_2,\gamma_2)) = \inf_{\varepsilon>0}\{F_1(B) \le F_2(B_\varepsilon) + \varepsilon, F_2(B) \le F_1(B_\varepsilon) + \varepsilon, \ \forall B \in \mathscr{B}^\infty\}$$

where $\mathscr{B}^\infty$ are the Borel sets in $\mathbb{R}^\infty$, B_ε is the open ε-neighborhood around B, and F_j is the measure on $\mathscr{B}^\infty$ defined by

$$F_j(T_1 \le t_1, T_2 \le t_2, \cdots, T_n \le t_n) = \gamma_j Q_j(t_1) Q_j(t_2) \cdots Q_j(t_n) U.$$

Since the Levy metric induces the topology of weak convergence, results such as Whitt [18] show that a large class of queues are

continuous operators in this metric. The problem is computing or approximately computing distances in the metric. If this can be done it might be possible to get bounds on the error when one MRP is substituted for another. One suspects that if two queues are identical except for the arrival streams, behavior of the queues will be "close" if the two arrival streams are "close" in the metric. For example consider two queues in tandem where the first is M/M/1/N and the second is $\cdot$/M/1. If $N < \infty$ the arrival process to the second queue is not a renewal process, and the steady state queue length at the second queue will not be geometric as it would be if $N = \infty$. For large N, though, one would expect to find little difference between the true steady state distribution and the geometric. (see example 2.4.3)

4. Abstractions

Some of the rich mathematical structure of the space of MRP's should be pointed out, although no attempt will be made here to exploit these ideas.

First of all, corresponding to each $N \leq \infty$ is the space of N state MRP's. This space can be further broken down to the space of N state MRP's with initial distribution γ, $\gamma \in \mathscr{P}_N$. The space of N state MRP's with initial distribution γ is a semigroup under at least two different forms of multiplication. The results of Chapter IV show that the space is a semigroup under convolution, and it is easy to verify that $(Q_1,\gamma)\cdot(Q_2,\gamma) = (Q_3,\gamma)$ where $Q_3(t) = Q_1(t)Q_2(t)$ is a multiplication. Both of these semigroups are noncommutitive.

The space of finite or countable state MRP's is also a metric space where $d((Q_1,\gamma_1), (Q_2\gamma_2)) = 0$ if $(Q_1,\gamma_1) \sim (Q_2,\gamma_2)$. By allowing more general state spaces, the space might be complete with respect to

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the metric. A reasonable conjecture is that the space of MRP's with state spaces in $\mathscr{B}^{\infty}$ is complete under the Levy metric. The topology of this space (or even the space of finite state MRP's) is interesting to think about since MRP's with vastly different state spaces can be "close" in the metric.

Perhaps the most fundamental observation is that the space of MRP's is a category. There are several morphisms between MRP's to choose from. Strong lumpability, weak lumpability, collapsibility and equivalence are all morphisms (theorem 2.6.6). Consider the finite and countable state MRP's. Let $\mathscr{E}$ be a map from the category of MRP's to the category of matrix rings defined by

$$\mathscr{E}\{(Q,\gamma)\} = \mathcal{Q}$$

where $\mathcal{Q}$ is the matrix ring generated by $\{Q(t)\}$, $t \in [0,\infty]$. Equivalence between two MRP's is too weak to induce a functor since it is possible for $(Q,\gamma) \sim (Y,\beta)$ without a homomorphism between $\mathcal{Q}$ and $\mathcal{Y}$. Theorems 2.5.1 and 2.6.5 show that $\mathscr{E}$ is a functor if the morphism used is strong lumpability, weak lumpability or collapsibility. In fact, collapsibility occurs between two MRP's if and only if there is a certain homomorphism between the rings. Thus, from this point of view, collapsibility is the most pleasing morphism of the four. Although strong and weak lumpability imply homomorphisms, the converse is not necessarily true. It is important to note that weak lumpability, collapsibility and equivalence are identical in the special case where one of the MRP's is a renewal process. It has still not been formally shown that collapsibility and weak lumpability are different, although in the finite or countable case, weak lumpability clearly implies collapsibility, and in view of (2.6.1) and (2.6.2) the converse is doubtful.

Let us examine these morphisms in a general setting. Let (Q,μ) be a MRP with state space $(S_1, \mathscr{S}_1)$ and let (Y,λ) be a MRP with state space $(S_2, \mathscr{S}_2)$. Let Φ be a measurable map of S_1 onto S_2. Say (Q,μ) is $\{X_n, T_n\}$ and (Y,λ) is $\{Z_n, V_n\}$.

Since (Q,μ) is a MRP,

$$P(X_{n+1} \in A,\ T_{n+1} \le t \,|\, X_n = s) = Q_{sA}(t)$$

is a regular conditional probability on $\mathscr{S}_1$. Assume that $P(X_0 \in A|\Phi)(s)$ is also a regular conditional probability on $\mathscr{S}_1$. Since

$$\forall\, s \in \Phi^{-1}(u),\ P(X_0 \in A|\Phi)(s) = P(X_0 \in A|X_0 \in \Phi^{-1}(u))$$

we can denote $P(X_0 \in A|\Phi)(s)$ by $P(u,A)$ where $u = \Phi(s)$. $P(u,\cdot)$ is a measure on $\mathscr{S}_1$ for each $u \in \mathscr{S}_2$, and $P(\cdot,A)$ is $\mathscr{S}_2$-measurable for every $A \in \mathscr{S}_1$. $P(u,A)$ is analogous to Π in section 2.6.

Let $\mathscr{Q}$ be the set of all maps $Q\colon S_1 \times \mathscr{S}_1 \to \mathbb{R}$ such that for each $s \in S_1$, $Q(s,\cdot)$ is a (signed) measure on $\mathscr{S}_1$ and for each $A \in \mathscr{S}_1$, $Q(\cdot,A)$ is $\mathscr{S}_1$-measurable. Likewise let $\mathscr{Y}$ be the set of all such maps $Y\colon S_2 \times \mathscr{S}_2 \to \mathbb{R}$. Define addition in $\mathscr{Q}$ and $\mathscr{Y}$ to be pointwise addition and define multiplication to be

$$Q_1Q_2(s,A) = \int Q_1(s,dx)Q_2(x,A).$$

Let $\mathcal{Q} \subset \mathscr{Q}$ be the ring generated by $\{Q(t)\}$ and let $\mathcal{Y} \subset \mathscr{Y}$ be the ring generated by $\{Y(t)\}$. Define the map $\Xi : \mathcal{Q} \to \mathcal{Y}$ to be

$$\Xi(Q)_{uB} = \int Q(s, \Phi^{-1}(B))P(u,ds).$$

The morphisms can now be defined in the general setting.

<u>Definition</u> 5.4.1. Q is strongly lumpable to Y via Φ if for every probability measure μ on $(S_1, \mathscr{S}_1)$, $\{\Phi(X_n), T_n\}$ is a MRP on $(S_2, \mathscr{S}_2)$ with kernel $Y(t)$.

Definition 5.4.2. Q is weakly lumpable to Y via Φ if for some probability measure μ on $(S_1, \mathscr{S}_1)$, $\{\Phi(X_n), T_n\}$ is a MRP on $(S_2, \mathscr{S}_2)$ with kernel Y(t).

Definition 5.4.3. (Q,μ) is collapsible to Y via Φ if Ξ is a homomorphism.

Definition 5.4.4. (Q,μ) is equivalent to (Y,λ) if $\{T_n\}$ and $\{V_n\}$ have the same distribution.

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